Chapter 2. Exponential and Log functions

This material is in Chapter 6 of Anton Calculus.

The basic idea here is mainly to add to the list of functions we know about (for calculus) and the ones we will study all have applications. What we know so far are basically algebraic functions (like polynomials built out of positive integer powers x^n, for example $f(x) = -1 + 3x - 5x^2 + 17x^3$, more generally rational functions that are quotients $f(x) = p(x)/q(x)$ of polynomials $p(x)$ and $q(x)$ where $q(x)$ is not identically zero) and trigonometric functions $\sin x$, $\cos x$, $\tan x$, etc.

We do also know about roots like $\sqrt{x}$ and $x^{1/3}$, which are examples of inverse functions (and also count as algebraic). Finally we can combine all these by adding, multiplying or applying one after the other (which is composition of functions $g(f(x))$). We have enough rules to differentiate all these — basic rules for derivatives of x^n, x^r with $r \in \mathbb{Q}$ a rational, $\cos x$, $\sin x$ and so on, the rules for differentiating sums and constant multiples, the product rule, the quotient rule and finally the chain rule.

We can differentiate most any $f(x)$ we can write down by a formula, as long as it makes sense for some intervals of x (and sometimes there are bad points like $x = 0$ for $f(x) = |x|$ or $f(x) = x^{1/3}$ to avoid).

However, life is more complicated for finding antiderivatives. We need more functions to cope with some quite easy-looking integrals. And the functions we study in this chapter have practical uses.

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2.1 Exponential function

(More or less as in Anton §6.1.)

One way to define the *exponential function* $f(x) = \exp(x) = e^x$ is to say that it is the only function $f \colon \mathbb{R} \to \mathbb{R}$ with the properties

$$f'(x) = f(x) \text{ and } f(0) = 1.$$

It is perhaps not a good way to define it but it is a valid way.

What it means is that the function $y = e^x$ solves a *differential equation*

$$\frac{dy}{dx} = y$$

We will explain a little later what the number e is and why the nottaion e^x is used. For now we will use $f(x)$ or $\exp(x)$.

The drawback of the approach here is how to know that there is a solution of the differential equation (apart from $f(x) = 0$ which won't satisfy $f(0) = 1$). Leaving this aside until a little later, it is not hard to check some properties it must have (if there is such a function).

- $\frac{d}{dx}(f(x)f(-x)) = f'(x)f(-x) + f(x)f'(-x)(-1) = f(x)f(-x) - f(x)f(-x) = 0.$
- So $f(x)f(-x)$ is constant.[1] Looking at $x = 0$ we find that the constant is $f(0)f(0) = 1$.
- From $f(x)f(-x) = 1$ for all x, we can conclude that $f(x)$ is never 0.
- It follows that $f(x) > 0$ for all x.

  Proof. This is not that easy to establish carefully. It really needs the Intermediate Value theorem (discussed in the book, \$1.5). That theorem concerns well-behaved functions on intervals (called continuous functions) that are positive somewhere and negative somewhere else and the conclusion is that the function must be zero in between. Since $f(1) = 1 > 0$, our function could never be negative.[2] Graphically, it is sort of obvious that a graph can't get from below the horizontal axis to above, without crossing — but if there was a gap in the domain, it could. □

- Since $f'(x) = f(x) > 0$ for all x, it follows that $f(x)$ is strictly increasing on $\mathbb{R} = (-\infty, \infty)$.[3]

  In particular $1 = f(0) < f(1)$.

[1]This really needs the Mean Value Theorem for derivatives, discussed in §3.8 of Anton's book. It is hard to think of anything but a constant function that has derivative always zero. As long as the domain has no gaps, there are no others.

[2]Differentiable functions such as our $f(x)$ are continuous.

[3]Again you are used to this, but a correct proof is not so easy. It follows from the Mean Value Theorem.

- For $r \in \mathbb{Q}$,

$$\begin{aligned}\frac{d}{dx} f(rx)(f(x))^{-r} &= rf'(rx)(f(x))^{-r} + f(rx)(-r)(f(x))^{-r-1}f'(x) \\ &= rf(rx)(f(x))^{-r} - rf(rx)(f(x))^{-r-1}f(x) = 0\end{aligned}$$

  can be checked.

  So
$$f(rx)(f(x))^{-r} = \text{constant}.$$

  For $x = 0$ we get
$$f(0)(f(0))^{-r} = \text{constant}$$
  and since $f(0) = 1$, this tell us that the constant is 1. So
$$f(rx)(f(x))^{-r} = 1 \qquad (\text{all } x \in \mathbb{R}).$$

  Thus $f(rx) = (f(x))^r$ (for $r \in \mathbb{Q}$).

- In particular $f(r) = f(1)^r$ for $r \in \mathbb{Q}$.

  We **define the number** e to be $e = f(1)$ and **define** e^x by
$$e^x = f(x).$$

  Note that for $x \in \mathbb{Q}$ (where we had a sensible way to define e^x previously) this is no change.

  Another notation that is commonly used is $\exp(x) = f(x)$.

- With this definition, we have the law of exponents $e^{x+a} = e^x e^a$ valid for all $x, a \in \mathbb{R}$.

  Proof. Look at
$$g(x) = \frac{e^{x+a}}{e^x} = \frac{f(x+a)}{f(x)} = f(x+a)f(-x)$$
  (with a fixed) and check that $g'(x) = 0$. So $g(x)$ is constant. But $g(0) = f(a)$ and so $g(x) = f(a)$ for all x. Then we have $f(x+a)/f(x) = f(a)$ or $f(x+a) = f(x)f(a)$, as we wanted. □

- We can compute
$$\frac{d}{dx}\frac{e^x}{x} = \frac{e^x x - e^x}{x^2} = \frac{e^x(x-1)}{x^2} > 0 \text{ for } x > 1$$

  So e^x/x is (strictly) monotone increasing for $x > 1$. In particular, for $x > 2$
$$\frac{e^x}{x} > \frac{e^2}{2}$$

and that means that with $c = e^2/2$ we have

$$e^x > cx \quad (x > 2).$$

As $c > 0$, we can say that $f(x) = \exp(x)$ becomes larger and larger with x.

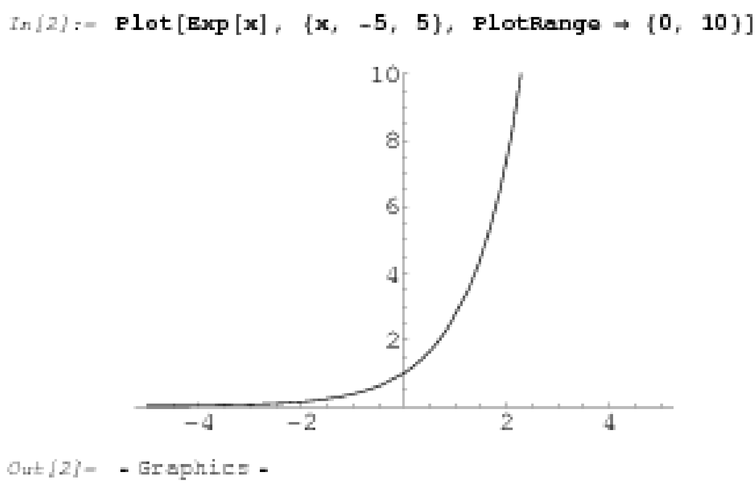
Using a notation you saw in Anton §1.3,

$$\lim_{x\to\infty} e^x = \infty$$

- Since $e^{-x} = 1/e^x$, we can then conclude that e^x becomes very small as x becomes large and negative. So

$$\lim_{x\to-\infty} e^x = 0$$

- Since we now know $e^x > 0$ always, e^x is strictly increasing and taking account of the two previous points, we can say that the graph of $y = e^x$ must look something like this



We have not come up with any way to estimate the value of $e = f(1)$, which influences the graph. The graph seems to run off the page quite quickly to the right and merge into the x-axis to the left. This is something we can check, at least roughly.

- For $n \geq 1$ we can compute that

$$\frac{d}{dx}\frac{e^x}{x^n} = \frac{e^x x^n - e^x(nx^{n-1})}{x^{2n}} = \frac{e^x(x-n)}{x^{n+1}} > 0 \text{ for } x > n$$

So e^x/x^n is (strictly) monotone increasing for $x > n$. In particular, for $x > n+1$

$$\frac{e^x}{x^n} > \frac{e^{n+1}}{(n+1)^n}$$

and that means that with $c_n = e^{n+1}/(n+1)^n$ we have

$$e^x > c_n x^n \quad (x > n+1).$$

So, for $n > 1$ and $x > n+2$ we have

$$\frac{e^x}{x^n} > \frac{c_{n+1} x^{n+1}}{x^n} = c_{n+1} x$$

and that implies

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$$

(since $c_{n+1} > 0$).

Conclusion: in the long run e^x grows faster that any power x^n as $x \to \infty$. Or '*exponential growth is fast!*'.

All this discussion has the drawback that we assumed there was some $f(x)$ defined for all $x \in \mathbb{R}$ and satisfying the properties $f'(x) = f(x)$ and $f(0) = 1$. In §6.1 in Anton, he says that if b^r makes sense for $b > 0$ and $r \in \mathbb{Q}$ (which we know), there must be a way to make sense of b^x for all $x \in \mathbb{R}$. And then there is a magic number e so that $f(x) = e^x$ satisfies the properties we want.

Another approach, only slightly less mysterious, is to notice that the polynomials

$$p_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

(where $n!$ is pronounced n *factorial* and means the product of the numbers n, $n-1$, …, 2, 1) nearly do what we want because

$$\begin{aligned} p_n'(x) &= 0 + 1 + 2\frac{x}{2!} + 3\frac{x^2}{3!} + \cdots + +n\frac{x^n}{n!} \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} \\ &= p_{n-1}(x) \end{aligned}$$

It is in fact true that $x^n/n!$ is small (for a given x) if n is large and so the difference between $p_n'(x)$ and $p_n(x)$ looks small. If we take

$$f(x) = \lim_{n\to\infty} p_n(x) = \lim_{n\to\infty}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right),$$

it is actually possible to prove that $f'(x) = f(x)$ and is is really obvious that $f(0) = 1$.

So this might be a better (less mysterious) starting point for $f(x) = \exp(x)$. It is a well-known formula for e^x, called the Taylor series formula. We'll come back to series in Chapter 9 of Anton, and then this kind of formula will seem more normal.

2.1.1 Example (Exponential growth/decay). There are several realistic situations where a differential equation

$$\frac{dy}{dx} = ky$$

arises. In some examples, x is time and so we might be better with

$$\frac{dy}{dt} = ky \tag{1}$$

where now $y = y(t)$ depends on t. The examples include population growth (say of flies, or animals, where there is no food shortage or environmental change) where $y(t)$ is the number in the population at time t and k is the difference between the birth rate and the death rate (per unit time). Another would be radioactive decay, where $y(t)$ is the number of radioactive atoms in a sample at time t and $|k|$ is the proportion that will decay (and then stop being radioactive) per unit time. Here $k = -|k| < 0$. In these examples, we need to assume the number is large so that considering $y(t)$ to vary continuously (and not just through integer values) is not a big concern. Another example involves compound interest, where the compounding is done continuously (or over very small time steps). Another example, Newton's law of cooling, is mentioned in Anton §6.4, where the unknown function $y(t)$ is the difference between the temperature of an object at time t and the ambient temperature (though this interpretation of the law involves some simplifying assumptions).

To see what the solution of (1) are we rewrite it as

$$\frac{dy}{dt} - ky = 0$$

and then multiply across by e^{-kt} to get

$$e^{-kt}\frac{dy}{dt} - ke^{-kt}y = 0$$

The point of this is that now the left hand side is (by the product rule) the derivative of a product. So we get

$$\frac{d}{dt}(e^{-kt}y) = 0.$$

The factor e^{-kt} is known as an '*integrating factor*' in this situation. (This idea we use here is discussed more generally in Anton §8.4.) We conclude

$$e^{-kt}y = c = \text{constant}$$

and so $y = ce^{kt}$ gives all possible solutions.

Assuming y is a population, we need $c > 0$. If $k > 0$ the graph of y against t looks more or less similar to the graph $y = e^t$. The factor k in the exponent re-scales the t axis by $1/k$ while the c re-scales the y-axis.

If $k < 0$, then we need to reflect the exponential graph in the y-axis (and rescale the axes).

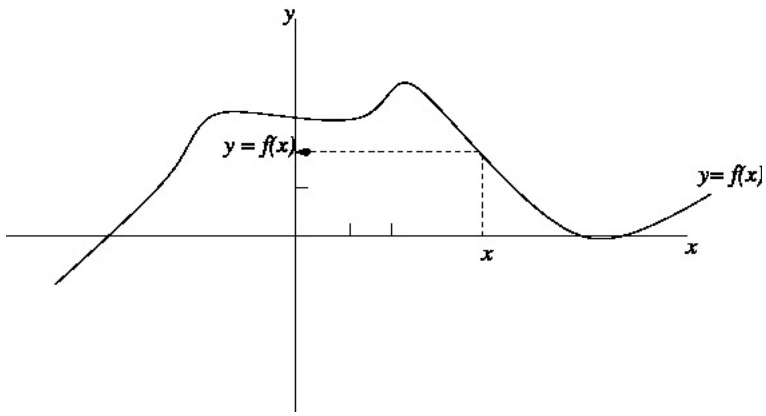
2.1.2 Exercise. Using an integrating factor trick, find all the solutions of the differential equation

$$\frac{dy}{dx} - 3y = e^{2x}.$$

2.2 Inverse functions

(Recall Anton §0.4 and see Anton §6.3.)

In this context it is useful to think of a function $y = f(x)$ as a machine (or black box) that does something predictable to an input x to produce an output y. Recall the diagram



from Anton §0.4.

The inverse function is supposed to undo what the function does (or do the reverse). So the inverse function should send y back to x, if the function sends x to y. (Think of reversing the arrows.)

You may also realise that the q^{th} root function $y \mapsto y^{1/q}$ is a kind of inverse of the q^{th} power function $x \mapsto x^q$ (or $y = x^q$). This is no problem for odd q (because then x^q is increasing on the whole range $-\infty < x < \infty$), but it is a problem for even q. The problem is that many values of y came from more than one x (two x's in fact) and so it is not clear how to choose which x to send such a y back to.

What we need in order to have an inverse function is that no horizontal line crosses the graph more than once. So the picture above does not have an inverse function and neither does $y = x^2$ (or $y = x^4$ or $y = x^6$) have a genuine inverse. When we define the square root function to be the positive square root, we are (rather arbitrarily) deciding to cut down the function $y = x^2$ to the domain $x \geq 0$. The cut-down function then has an inverse.

Here is a handy fact. A *monotone function* is one that is **either** increasing **or** descreasing.

2.2.1 Proposition. *If $A \subset \mathbb{R}$ and $f\colon A \to \mathbb{R}$ is a strictly monotone function, then no horizontal line crosses the graph $y = f(x)$ more than once.*

Proof. Let's take the case where f is strictly monotone increasing. Say there are two points on the graph $y = f(x)$ which are also on the horizontal line $y = y_0$. That means there are two different points $x_1, x_2 \in A$ with $(x_1, f(x_1)) = (x_1, y_0)$ and $(x_2, f(x_2)) = (x_2, y_0)$. Since $x_1 \neq x_2$ we must have either $x_1 < x_2$ or $x_2 < x_1$. If $x_1 < x_2$, then $f(x_1) < f(x_2)$ by definition of what it means for f to be strictly monotone increasing. But that is not so as $f(x_1) = f(x_2) = y_0$. On the other hand if $x_2 < x_1$, then $f(x_2) < f(x_1)$ (which is again not so).

For the case where f is strictly monotone decreasing, we can repeat almost the same argument, with small changes at the end. □

There is in fact a theorem that says that the converse of the proposition is true for certain nice functions on intervals (called continuous functions — differentiable functions are included).

We concentrate then on strictly monotone functions $f\colon A \to \mathbb{R}$ with A an interval. Usually the range $B = \{f(x) : x \in A\}$ will also be an interval. (Technically there is a theorem that if $f(x)$ is continuous on the interval A, then B will be an interval.) The inverse function will be the function $g(y)$ with domain B given by the rule

$$g(y) = x \text{ exactly when } y = f(x).$$

It is usual to write f^{-1} rather than g for the inverse function. So we get

$$f^{-1}(y) = x \text{ exactly when } y = f(x).$$

2.2.2 Theorem. *If $y = f(x)$ is a function defined on an interval A and if $f'(x) > 0$ for all $x \in A$ then $x = f^{-1}(y)$ is differentiable on its domain and has derivative*

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}.$$

If instead $f'(x) < 0$ for all $x \in A$, then the same conclusion holds.

This rule is perhaps easier to remember as

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{\left(\frac{dy}{dx}\right)}.$$

We will not prove the theorem as it is quite tricky to prove it correctly. From implicit differentiation is is quite easy to see that that formula for dx/dy has to be right **IF** we already know that there is a derivative dx/dy for the inverse function.

2.2.3 Exercise. Check that the theorem gives the right result for the case of the function $y = x^q$ where we restrict $x > 0$ (and $q \in \mathbb{N}$).

2.3 The natural logarithm

(See Anton §6.1.)

The exponential function $y = e^x$ has strictly positive derivative for all $x \in (-\infty, \infty) = \mathbb{R}$. So it will have an inverse function. The domain of the inverse function will be $(0, \infty)$ according to our earlier considerations of the exponential (and its graph).

2.3.1 Definition. The natural logarithm function $\ln\colon (0, \infty) \to \mathbb{R}$ is the inverse of the exponential function.

2.3.2 Remark. So if we start with $y = e^x$ then

$$\ln y = x \text{ means exactly that } y = e^x.$$

Exchanging the roles of x and y we get

$$\ln x = y \text{ means exactly that } x = e^y.$$

Note that $\ln x$ only makes sense for $x > 0$.

Properties of the natural logarithm

(i) $\ln e = 1$ (because $e = e^1$).

(ii) $\frac{d}{dx}\ln x = \frac{1}{x}$ (for $x > 0$).

Proof. Write $y = \ln x$, so that $x = e^y$. That tells us

$$\frac{dx}{dy} = e^y$$

and so

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}. \qquad \square$$

(iii) For $a > 0$, $\ln(1/a) = -\ln a$.

Proof. Let $x = \ln a$ so that $e^x = a$. But then we know $e^{-x} = 1/e^x = 1/a$. So $-x = \ln(1/a)$. That says $-\ln a = \ln(1/a)$. $\square$

(iv) For $a, b > 0$, $\ln(ab) = \ln a + \ln b$.

Proof. Let $x = \ln a$ and $t = \ln b$. Then we know $e^x = a$ and $e^t = b$. From that we have $ab = e^x e^t = e^{x+t}$ and then

$$\ln(ab) = x + t = \ln a + \ln b. \qquad \square$$

(v) For $x > 0$ and $r \in \mathbb{Q}$, $\ln(x^r) = r \ln x$.

Proof. Put $y = \ln x$ so that $e^y = x$. Then we know $(e^y)^r = e^{ry}$ (from properties of the exponential) and so $x^r = e^{ry}$ — which means exactly that $\ln(x^r) = ry = r \ln x$. $\square$

(vi) The graph of the natural logarithm function is obtained by reflecting the graph of the exponential in the diagonal line $y = x$. That is because the graph of $y = \ln x$ is the same as the graph of $e^y = x$.

2.4 Arbitrary powers of positive numbers

(See Anton §6.1.)

We were able to prove above that $\ln(x^r) = r \ln x$ for $x > 0$ and $r \in \mathbb{Q}$. Why did we not prove it for $r \in \mathbb{R}$?

The answer is that we have not yet defined x^a for $x > 0$ and $a \in \mathbb{R}$. We did define $e^a = \exp(a)$, and in that we thought we were on safe ground because $e^r = \exp(r)$ for $r \in \mathbb{Q}$.

For $x > 0$ we do know

$$x = e^{\ln x}$$

(because that is what inverse functions do — one undoes what the other does). So for $r \in \mathbb{Q}$ we do know

$$x^r = \left(e^{\ln x}\right)^r = e^{r \ln x}.$$

Taking this as a guide we can make it into a definition for arbitrary powers, not just rational powers.

2.4.1 Definition. For $x > 0$ and $a \in \mathbb{R}$ we define

$$x^a = e^{a \ln x} \qquad \text{(which may be easier to remember as } x^a = (e^{\ln x})^a = e^{a \ln x}\text{).}$$

2.4.2 Proposition. *With this definition the laws of exponents hold: For $x, y > 0$ and $a, b \in \mathbb{R}$ we have*

(i) $1/(x^a) = x^{-a}$

(ii) $(x^a)^b = x^{ab}$

(iii) $x^a x^b = x^{a+b}$

(iv) $(xy)^a = x^a y^a$

This is not hard to prove based on the above definition and the properties of the exponential that we already know.

2.4.3 Examples. i) For $x > 0$ and $a \in \mathbb{R}$

$$\frac{d}{dx} x^a = \frac{d}{dx}\left(e^{\ln x}\right)^a = \frac{d}{dx} e^{a \ln x} = e^{a \ln x} a \frac{1}{x} = a x^a x^{-1} = a x^{a-1}.$$

ii) For $a > 0$ and $x \in \mathbb{R}$

$$\frac{d}{dx} a^x = \frac{d}{dx}\left(e^{\ln a}\right)^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \ln a = (\ln a) a^x.$$

2.4.4 Exercise. Find $\dfrac{d}{dx} x^x$.

2.5 Inverse trigonometric functions

(See Anton §6.7.)

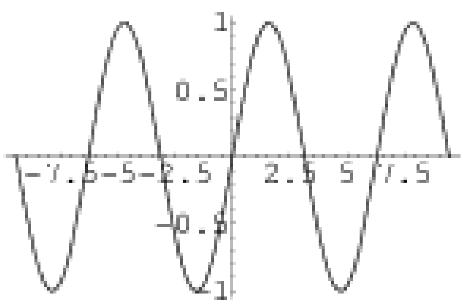
The trigonometric functions $y = \sin x$, $y = \cos x$, $y = \tan x$ (and so on) do not have any inverse in the ordinary way because horizontal lines often cut their graphs many times. In fact they are all periodic with period 2π (and $\tan x$ has period π), which means that

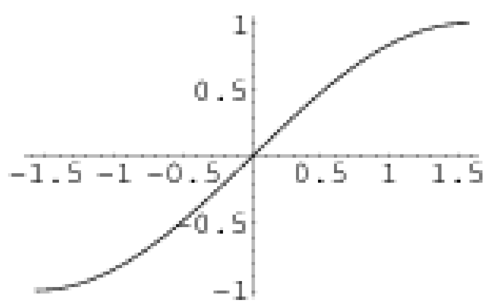
$$\sin(x + 2\pi) = \sin x, \quad \cos(x + 2\pi) = \cos x, \quad \tan(x + \pi) = \tan x.$$

So if we know $y = \sin x$, there is no way to know what x is, no inverse function.

We've seen this before. The function $y = x^2$ has no inverse either, but we came up with $\sqrt{y}$ by taking the positive square root. That is probably unfair discrimination against negative numbers, but it seems handy to have a square root function.

We do even more drastic things to come up with 'inverse' trigonometric functions. Here is the graph of $y = \sin x$ (for $-3\pi \leq x \leq 3\pi$) and a graph of a very much restricted $y = \sin x$, restricted to $-\pi/2 \leq x \leq \pi/2$.





The restricted graph is a strictly monotone increasing graph and when we write $\sin^{-1} y$ we mean the inverse function of this cut down $y = \sin x$. (Some people use the notation $\arcsin y$ instead of $\sin^{-1} y$. That emphasises that it is not really an inverse function.)

In summary

$$x = \sin^{-1} y \text{ means exactly that } \sin x = y \text{ and } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.$$

Changing around the names of the variables, we get

2.5.1 Definition. The '*inverse* $\sin$ *function*' (or $\arcsin$ function) is the function $\sin^{-1} \colon [-1, 1] \to [-\pi/2, \pi/2]$ given by

$$\theta = \sin^{-1} x \text{ means exactly that } \sin \theta = x \text{ and } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

2.5.2 Proposition. *The derivative of* $y = \sin^{-1} x$ *is*

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}} \qquad (-1 < x < 1).$$

Proof. If $y = \sin^{-1} x$, then $\sin y = x$. Taking d/dx of both sides, we get

$$\cos y \frac{dy}{dx} = 1$$

and so

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

(We could also use the result on derivatives of inverse functions to get the same thing.) We want the answer in terms of x and we use

$$\cos^2 y + \sin^2 y = 1 \Rightarrow \cos^2 y = 1 - \sin^2 y \Rightarrow \cos y = \sqrt{1 - \sin^2 y}$$

(which is true because $\cos y \geq 0$ when $-\pi/2 \leq y \leq \pi/2$). So

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}.$$

We must exclude $x = \pm 1$ to avoid division by 0. □

2.5.3 Note. Earlier, we learned that $\sin^2 x$ means $(\sin x)^2$. Now we see that $\sin^{-1} x$ has nothing to do with a power of $\sin x$. Rather it means the 'inverse' function. One might argue that the notation is potentially misleading!

2.5.4 Exercise. What is the graph of $y = \sin^{-1} x$?

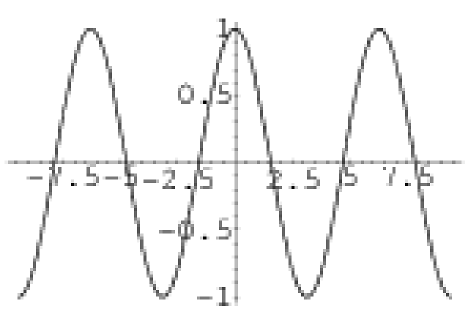
2.5.5 Definition. The '*inverse* $\cos$ *function*' (or $\arccos$ function) is the function $\cos^{-1} \colon [-1, 1] \to [0, \pi]$ given by

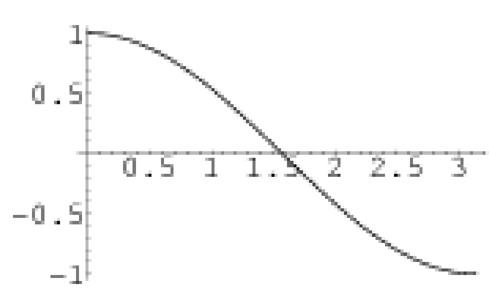
$$\theta = \cos^{-1} x \text{ means exactly that } \cos\theta = x \text{ and } 0 \leq \theta \leq \pi.$$

2.5.6 Proposition. *The derivative of* $y = \cos^{-1} x$ *is*

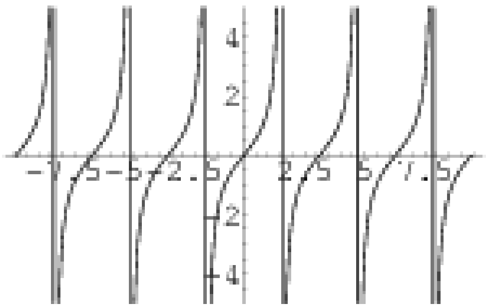
$$\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1 - x^2}} \qquad (-1 < x < 1).$$

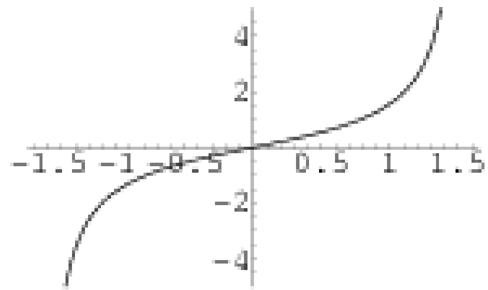
The proof is very similar to the one for $(d/dx)\sin^{-1} x$. Here is the graph of $y = \cos x$ $(-3\pi \leq x \leq 3\pi)$ and the restricted part of which we take the inverse function.





Here is the graph of $y = \tan x$ ($-3\pi \leq x \leq 3\pi$) along with the section $-\pi/2 < x < \pi/2$. The apparently vertical parts of the graph of $y = \tan x$ should not be there at all. (That is because $\tan x$ is not defined for x an odd multiple of $\pi/2$.)





The section for $-\pi/2 < x < \pi/2$ is strictly monotone increasing and has an inverse function.

2.5.7 Definition. The '*inverse* $\tan$ *function*' (or $\arctan$ function) is the function $\tan^{-1} \colon \mathbb{R} \to (-\pi/2, \pi/2)$ given by

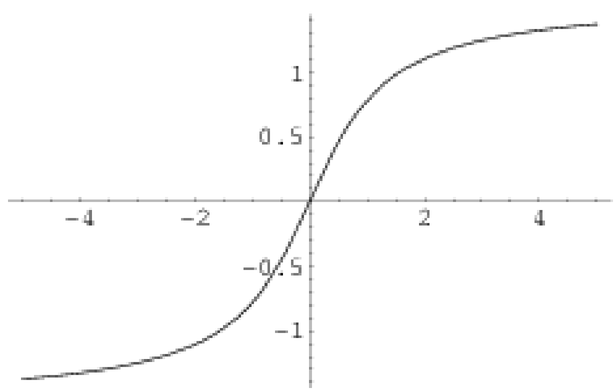
$$\theta = \tan^{-1} x \text{ means exactly that } \tan\theta = x \text{ and } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

2.5.8 Proposition. *The derivative of* $y = \tan^{-1} x$ *is*

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} \qquad (x \in \mathbb{R}).$$

The proof is again not that different (but relies on $1 + \tan^2\theta = \sec^2\theta$).

Here is the graph $y = \tan^{-1} x$.



2.6 Hyperbolic functions

(See Anton §6.8.)

2.6.1 Definition. The *hyperbolic cosine function* is denoted $\cosh x$ and is defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

2.6.2 Definition. The *hyperbolic sine function* is denoted $\sinh x$ and is defined by

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

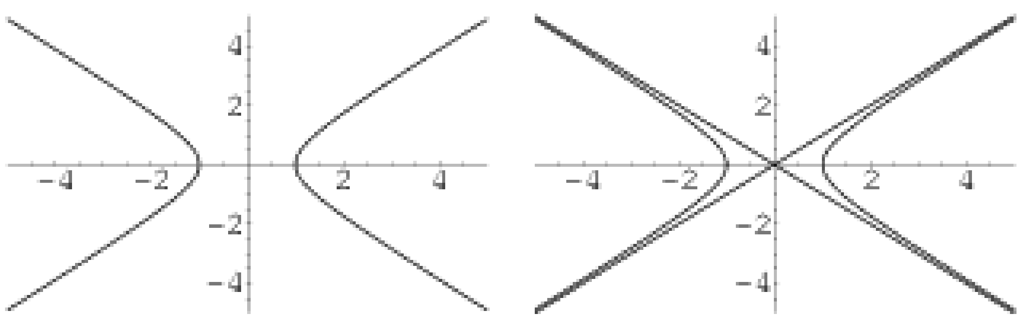
2.6.3 Proposition. *For every $x \in \mathbb{R}$,*

$$\cosh^2 x - \sinh^2 x = 1$$

Proof.

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{1}{4}(e^{2x} + e^{-2x} + 2e^x e^{-x} - (e^{2x} + e^{-2x} - 2e^x e^{-x})) \\ &= e^x e^{-x} = e^{x-x} = e^0 = 1 \qquad \square\end{aligned}$$

2.6.4 Remark. While $(\cos\theta, \sin\theta)$ lies on the unit circle, the above says that $(\cosh t, \sinh t)$ lies on the *standard hyperbola* $x^2 - y^2 = 1$. Here is the graph of that hyperbola, and a second version along with the two lines $y = \pm x$ (which are guiding lines that you can use to sketch the hyperbola). Since $\cosh t > 0$ always, $(\cosh t, \sinh t)$ lies on right half of the hyperbola. The value of t does not correspond to an angle in the picture.



2.6.5 Proposition.

$$\frac{d}{dx}\cosh x = \sinh x \quad \textit{and} \quad \frac{d}{dx}\sinh x = \cosh x.$$

2.6.6 Remark. A general principle is that for each formula about trig functions, there is a very similar formula for hyperbolic functions (but often with minus signs in different places). The hyperbolic functions are **not** periodic though.

2.6.7 Remark. It may be of interest to know that the graph of $y = \cosh x$ has the shape of a 'catenary', meaning the shape of a hanging chain (undisturbed by wind). By a 'chain' is meant something like a cable, but one that bends without reistance, yet still has mass.

2.7 Inverse hyperbolic functions

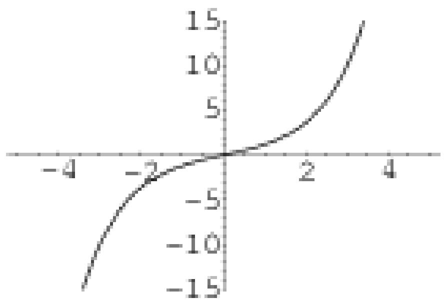
The hyperbolic sine function $y = \sinh x$ is strictly monotone increasing because $\frac{d}{dx} \sinh x = \cosh x = (e^x + e^{-x})/2 > 0$ always. So it has an inverse function. For $x > 0$ we have $\sinh x = (e^x - e^{-x})/2 > (e^x - 1)/2$ (since $e^{-x} < e^0 = 1$ for $x > 0$, or $-x < 0$). This shows that $\lim_{x\to\infty} \sinh x = \infty$. As $\sinh(-x) = -\sinh x$, we also have $\lim_{x\to-\infty} \sinh x = -\infty$.

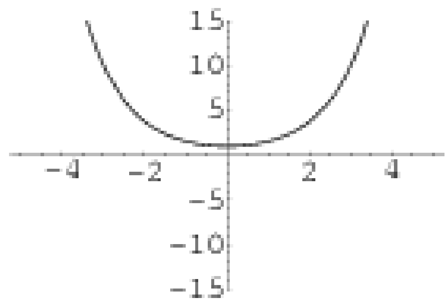
The hyperbolic cosine function $y = \cosh x$ is always positive. In fact $\cosh x \geq \cos 0 = 1$. We have $\frac{d}{dx} \cosh x = \sinh x$. For $x > 0$, we have $\sinh x > \sinh 0 = 0$ and so $\cosh x$ is strictly monotone increasing for $x > 0$. On the other hand, for $x < 0$, $\sinh x < \sinh 0$ and so $\cosh x$ is strictly monotone decreasing for $x < 0$. Also we have

$$\cosh x > \frac{1}{2} \max(e^x, e^{-x}) = \frac{1}{2} e^{|x|}$$

so that $\lim_{x\to\infty} \cosh x = \infty$ and also $\lim_{x\to-\infty} \cosh x = \infty$. In fact $\cosh x$ grows very rapidly, comparably fast to the exponential.

Here are graphs of $y = \sinh x$ and $y = \cosh x$.





The function $y = \sinh x$ has an inverse function $\sinh^{-1} \colon \mathbb{R} \to \mathbb{R}$. We can say then that

$$y = \sinh^{-1} x \text{ means exactly the same as } \sinh y = x$$

and the graph of $y = \sinh^{-1} x$ is the reflection of the graph of sinh in the line $y = x$. We can find dy/dx for $y = \sinh^{-1} x$ by the theorem on derivatives of inverse functions:

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{\cosh y}$$

and we can expresse that in terms of x using $\cosh^2 y - \sinh^2 y = 1$, $\cosh^2 y = 1 + \sinh^2 y = 1 + x^2$, to get

$$\frac{dy}{dx} = \boxed{\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1 + x^2}}}$$

(and it is right to have the square root because $\cosh y > 0$ always).

2.7.1 Proposition. *We can express* $\sinh^{-1} x$ *in terms of the natural logarithm as*

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

Proof. We can solve $\sinh y = x$ for y in terms of x as folows.

$$\begin{aligned} \frac{e^y - e^{-y}}{2} &= x \\ e^y - e^{-y} - 2x &= 0 \\ (e^y)^2 - 2xe^y - 1 &= 0 \end{aligned}$$

This is a quadradic equation for e^y. We get

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Since $\sqrt{x^2+1} > \sqrt{x^2} = |x|$, the minus sign would certainly make e^y negative — which is not possible. So we must have $e^y = x + \sqrt{x^2+1}$, and that means $y = \ln(x + \sqrt{x^2+1})$ as we required. □

For $\cosh^{-1}$, we have to interpret what we mean by the inverse because horizontal lines can cross the graph of $\cosh$ more than once.

2.7.2 Definition. By $\cosh^{-1}$ we mean the function $\cosh^{-1} \colon [1, \infty) \to [0, \infty)$ given by the rule

$$y = \cosh^{-1} x \text{ means } \cosh y = x \text{ and } y \geq 0.$$

We can find the derivative (as long as we don't go to the end point $x = 1$) in a similar way to the way we did above for $\sinh^{-1} x$. It is

$$\boxed{\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}} \, (x > 1).$$

We can also express $\cosh^{-1}$ via the natural logarithm as

$$\boxed{\cosh^{-1} x = \ln(x + \sqrt{x^2-1})} \, (x \geq 1).$$

2.7.3 Remark. There is an inverse for $y = \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. It is given by

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x} \qquad (-1 < x < 1),$$

and it has derivative

$$\frac{d}{dx} \tanh^{-1} x = \frac{1}{1-x^2} \qquad (-1 < x < 1).$$