An even simpler situation is where we take $f(x, y, z) \equiv 1$ to be the constant function 1. Then

$$f(x, y, z)\,dx\,dy\,dz = dx\,dy\,dz = \text{ volume } = dV$$

is just the volume of the little tiny piece. When we add these up (or integrate the constant function 1) we get

$$\iiint_D 1\,dx\,dy\,dz = \text{ total volume.}$$

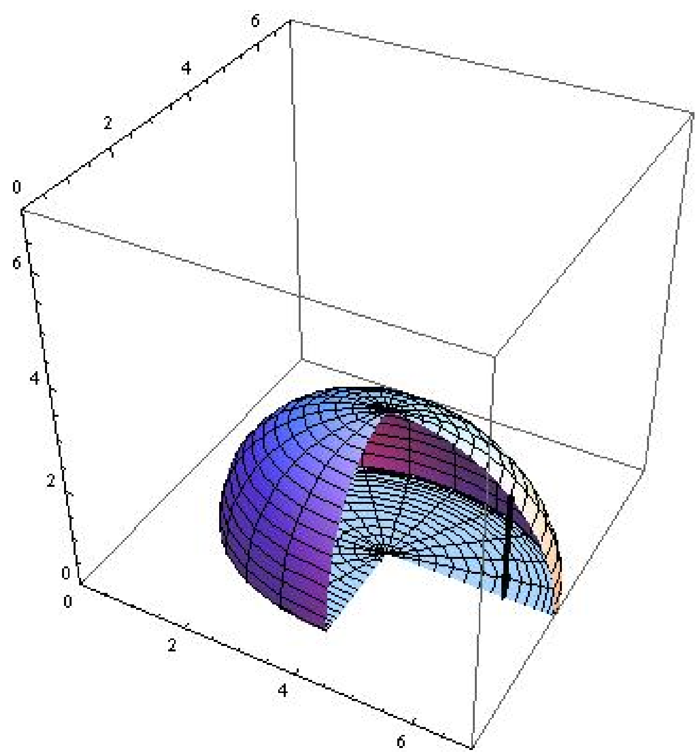
As for double integrals, there is a **Fubini Theorem** for triple integrals that allows us to work out $\iiint_D f(x, y, z)\,dx\,dy\,dz$ by working out three single integrals. The first integral (or inner integral) should be with respect to one of the variables, keeping the other two constant. Say we integrate dz first, keeping (x, y) fixed. We should integrate over all z that give points $(x, y, z) \in D$. At least in simple cases, that will be a range of z from a smallest we might call $z_0(x, y)$ to a largest $z_1(x, y)$.

In the following picture, there is an attempt to explain this. The curved surface is in fact the part of the ellipsoid

$$\frac{(x-4)^2}{3^2} + \frac{(y-2)^2}{2^2} + \frac{z^2}{3^2} = 1$$

where $z \geq 0$ and the flat part at the base is where $z = 0$. A quarter is cut away to allow us to see inside the object

$$D = \left\{(x, y, z) : \frac{(x-4)^2}{3^2} + \frac{(y-2)^2}{2^2} + \frac{z^2}{3^2} \leq 1, z \geq 0\right\}$$



For each fixed (x, y) (like the one shown) we have to find the limits for z so that $(x, y, z) \in D$ and it is easy enough to see that the smallest z is $z_0(x, y) = 0$, while the largest is

$$z_1(x, y) = 3\sqrt{1 - \frac{(x-4)^2}{3^2} - \frac{(y-2)^2}{2^2}}$$

obtained by solving for z in terms of (x, y) when (x, y, z) is on the upper (curved) surface.

We can see that the values of (x, y) for which there are any possible z to worry about are those (x, y) where the square root is a square root of something positive. So, those (x, y) where

$$\frac{(x-4)^2}{3^2} + \frac{(y-2)^2}{2^2} \leq 1.$$

Another way to think of it is that this is the outline of the object when viewed along the z-direction (from far away).

Anyhow our first integral (if we integrate dz first) is

$$\int_{z=0}^{z=3\sqrt{1-(x-4)^2/9-(y-2)^2/4}} f(x, y, z)\, dz$$

Say we next integrate dx, keeping y fixed. The inequality to be satisfied by all the (x, y) we need to worry about is above and it can be expressed as

$$-3\sqrt{1 - \frac{(y-2)^2}{2^2}} \leq x - 4 \leq 3\sqrt{1 - \frac{(y-2)^2}{2^2}}$$

so that our next integral should be

$$\int_{x=4-3\sqrt{1-(y-2)^2/4}}^{x=4+3\sqrt{1-(y-2)^2/4}} \left(\int_{z=0}^{z=3\sqrt{1-(x-4)^2/9-(y-2)^2/4}} f(x, y, z)\, dz \right) dx$$

Finally we have to integrate this dy. The limits for y are those corresponding to the extreme values of y for points in D. In this case the restriction on y is

$$1 - \frac{(y-2)^2}{2^2} \geq 0$$

and that turns out to be the same as

$$-2 \leq y - 2 \leq 2$$

or

$$0 \leq y \leq 4$$

So in this case $\iiint_D f(x, y, z)\, dx\, dy\, dz$ is

$$\int_{y=0}^{y=4} \left(\int_{x=4-3\sqrt{1-(y-2)^2/4}}^{x=4+3\sqrt{1-(y-2)^2/4}} \left(\int_{z=0}^{z=3\sqrt{1-(x-4)^2/9-(y-2)^2/4}} f(x, y, z)\, dz \right) dx \right) dy$$

It is possible to do the integrals in a different order. Say dy first, then dx and finally dz. All the limits will be changed if we do that and we would get

$$\int_{z=0}^{z=3} \left(\int_{x=4-3\sqrt{1-z^2/9}}^{x=4+3\sqrt{1-z^2/9}} \left(\int_{y=2-2\sqrt{1-(x-4)^2/9-z^2/9}}^{y=2+2\sqrt{1-(x-4)^2/9-z^2/9}} f(x,y,z)\,dy \right) dx \right) dz$$

An advantage of being able to choose the order is that sometimes the calculations are easier in one order than another. A trick that is sometimes useful for working out an iterated integral like the one we have just written is this:- figure out which $D \subset \mathbb{R}^3$ it corresponds to, write the integral as $\iiint_D f(x,y,z)\,dx\,dy\,dz$, that is use Fubini's theorem in reverse first, and then work out $\iiint_D f(x,y,z)\,dx\,dy\,dz$ with Fubini's theorem using a different order for the single integrals. Sometimes it turns a hard problem into one that is easier.

5 Change of variables in multiple integrals

We now come to a topic that goes by the name 'substitution' in the case of ordinary single integrals. You may recall that substitution for functions of one variable can be justified using the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \text{ when } y = y(u), u = u(x), y = y(u(x)).$$

For indefinite integrals it allows us to say that

$$\int g(u(x))\,dx = \int g(u)\frac{dx}{du}\,du$$

if we interpret the right hand side, a function of u, as a function of x by $u = u(x)$. For definite integrals we can change limits as well as variables and get an equation that says two numbers are equal.

$$\int_{x=a}^{x=b} g(u(x))\,dx = \int_{u=u(a)}^{u=u(b)} g(u)\frac{dx}{du}\,du.$$

The point to remember from this is that when we change from integrating over the interval $[a,b]$ in x, we must not only change the range of integration to the corresponding range in the u variable, we must also multiply the integrand by a factor dx/du.

For integrals in two variables (and similarly in three variables) we have to explain what that factor is that works in a similar way. It is the absolute value of a certain determinant of partial derivatives. Suppose we change from (x,y) to (u,v) say, we have to change $dx\,dy$ into a multiple of $du\,dv$ and the multiple is

$$\left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right| = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|$$

The matrix inside the determinant is called a *Jacobian matrix*. Its rows are the gradient vectors of x and y with respect to the u and v variables. The determinant itself is called a *Jacobian determinant*.

The rule is then that we must change $dx\,dy$ to

$$dx\,dy = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right| du\,dv$$

To change an integral

$$\iint_R f(x,y)\,dx\,dy$$

to an integral in $(u,v) = (u(x,y), v(x,y))$, we have to change R to the same set described in the (u,v) variables and change $dx\,dy$ as above.

For the case of triple integrals, if we change from (x,y,z) coordinates to

$$(u,v,w) = (u(x,y,z), v(x,y,z), w(x,y,z))$$

we have to make a similar change

$$dx\,dy\,dz = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}\right| du\,dv\,dw$$

While this theory can be applied to any change of coordinates, the ones that appear most often are polar coordinates in the plane (and somewhat related coordinate systems in space called cylindrical coordinates and spherical coordinates, neither of which we will discuss).

So we work out what these Jacobian factors are in polar coordinates. We can relate cartesian (x,y) coordinates in $\mathbb{R}^2$ to polar coordinates (r,θ) via

$$x = r\cos\theta, \quad y = r\sin\theta$$

and so we can work out the 4 partial derivatives we need for the Jacobian.

$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos\theta \\ \frac{\partial x}{\partial \theta} &= -r\sin\theta \\ \frac{\partial y}{\partial r} &= \sin\theta \\ \frac{\partial y}{\partial \theta} &= r\cos\theta \end{aligned}$$

Then the Jacobian matrix is

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

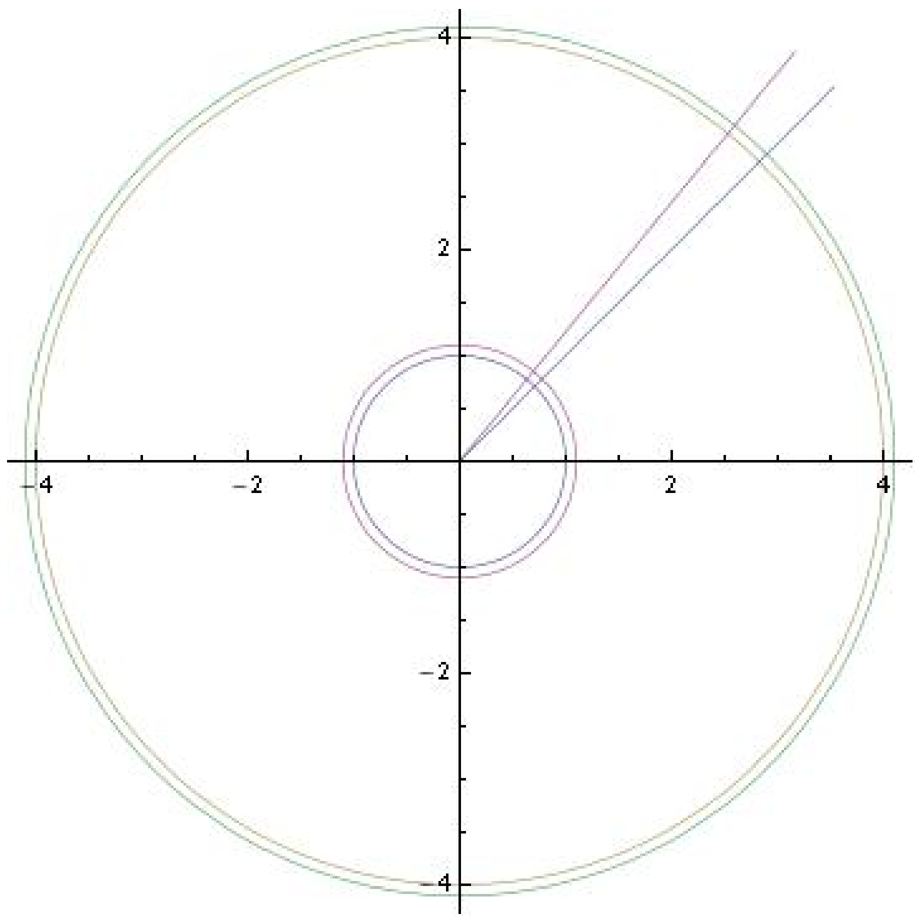
The determinant is

$$r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r.$$

The absolute value of the determinant is also r (as long as we keep $r > 0$ as we usually do for polar coordinates). This gives us the relation

$$\boxed{dx\,dy = r\,dr\,d\theta}$$

As we have not given any details on how these Jacobian factors are justified, we look at a picture relating to polar coordinates in the plane. The inner circle has radius 1, the next has radius 1.1, the next 4 and the outer one 4.1. The rays are at $\pi/4$ and $\pi/4 + 0.1$.



You can see that the area of the 'polar rectangle' at $(r, \theta) = (1, \pi/4)$ is smaller than the area of the polar rectangle at $(4, \pi/4)$. Both rectangles of squares of side 0.1 in polar coordinates, but it is reasonably clear that the outer one has 4 times the area of the inner one. In fact a polar rectangle with one corner at (r, θ) and opposite corner at $(r+dr, \theta+d\theta)$ is a (slightly bent) rectangle in the plane with side lengths dr and $r\,d\theta$. This is a way to see that the $r\,dr\,d\theta$ formula is at least plausible.

Example 5.1. Find

$$\iint_R \left(8 - \frac{(x^2+y^2)^2}{2}\right) dx\, dy$$

where $R = \{(x,y) : x^2 + y^2 \le 2\}$.

This is a problem that works out rather more easily in polar coordinates. We can describe R in polar coordinates as the points (r,θ) with $0 \le r \le \sqrt{2}$ and $0 \le \theta < 2\pi$. In this way we can express the integral as

$$\iint \left(8 - \frac{(r^2)^2}{2}\right) r\, dr\, d\theta$$

Putting in the limits

$$\begin{aligned}
\int_{\theta=0}^{\theta=2\pi} \left(\int_{r=0}^{r=\sqrt{2}} \left(8 - \frac{(r^2)^2}{2}\right) r\, dr \right) d\theta &= \int_{\theta=0}^{\theta=2\pi} \left(\int_{r=0}^{r=\sqrt{2}} 8r - \frac{r^5}{2}\, dr \right) d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \left[4r^2 - \frac{r^6}{12} \right]_{r=0}^{r=\sqrt{2}} d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \left(8 - \frac{8}{12} \right) d\theta \\
&= \frac{23}{3}(2\pi) = \frac{46\pi}{3}
\end{aligned}$$