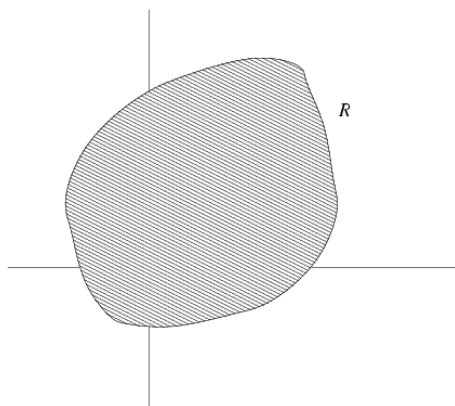


### 3 Double integrals

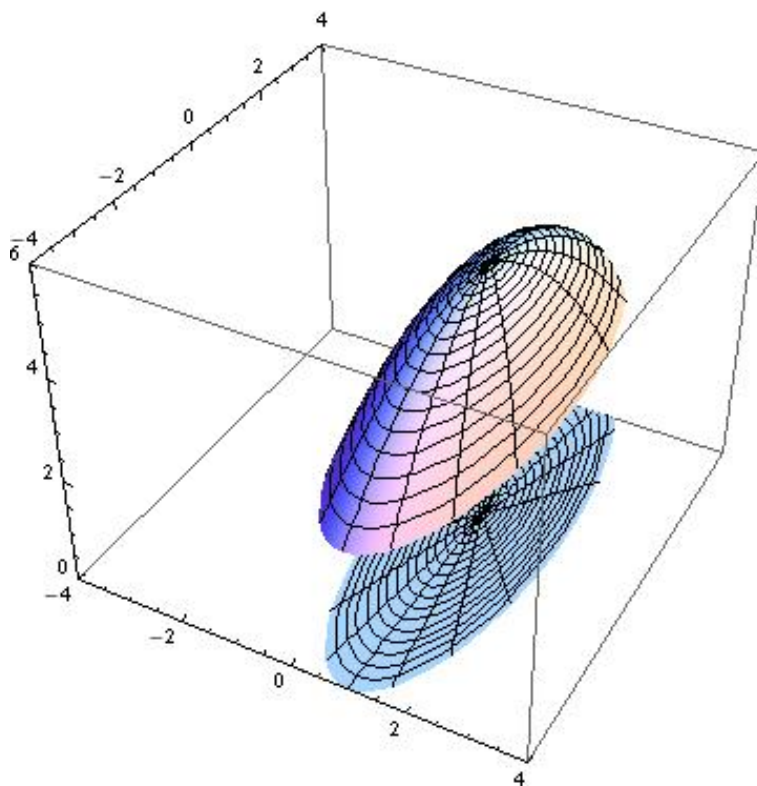
We are now going to give a brief definition of what a double integral is. We write double integrals as

$$\iint_R f(x, y) \, dx \, dy$$

where  $f(x, y)$  is a function of two variables that makes sense for  $(x, y) \in R$ , and  $R$  is a part of the  $(x, y)$ -plane. We should not allow  $R$  to be too complicated and we might picture  $R$  as something like this:



We could visualise the situation via the graph  $z = f(x, y)$  over the part of the horizontal plane indicated by  $R$ .



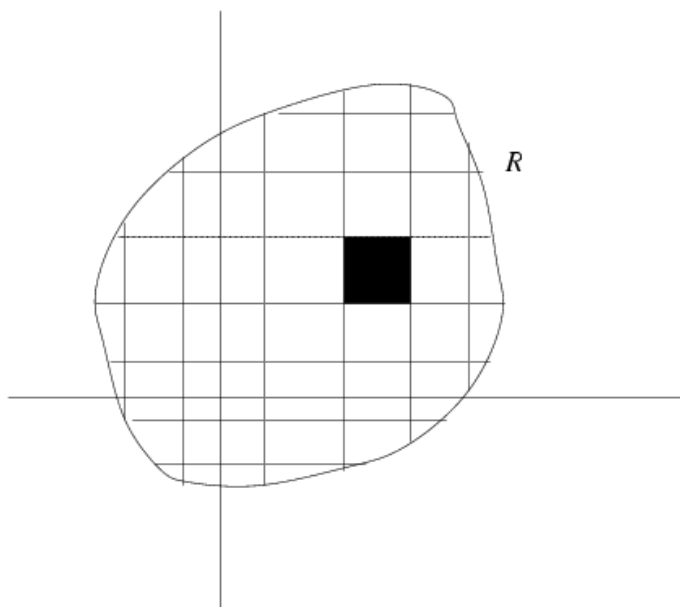
At least in the case where  $f(x, y) \geq 0$  always, we can imagine the graph as a roof over a floor area  $R$ . The graphical interpretation of the double integral will be that it is the volume of the part of space under the roof. (So think of a wall around the perimeter of the floor area  $R$ , reaching up to the edge of the ‘roof’ or graph. At least in this picture the solid part of space enclosed by the floor, the walls and the roof would look like some sort of irregular cake.)

So this is the version for double integrals of the explanation of ordinary single (definite) integrals  $\int_a^b f(x) dx$  as the ‘area under the graph’. To make it more accurate, we have to cater for functions  $f(x, y)$  that are sometimes negative, so that the ‘roof’ or graph is below the floor at times! In this case we have to subtract the volumes where the roof is below the floor, add the parts where the roof is above (that is the parts where  $f(x, y) > 0$  are counted positive, and the parts where  $f(x, y) < 0$  are counted negative).

So far what we have is a graphical or intuitive explanation, but it is not very useful if we want to compute the double integral. It is also unsatisfactory because it assumes we know what we mean by the volume and part of the reason for being at a loss for a way to compute the volume of such a strange ‘cake’ is that we don’t have a very precise definition of what the volume is.

For single integrals this ‘precise’ explanation proceeds by the idea of a limit, a limit of certain finite sums that seem to be plausible approximations to the idea we want to have.

To sketch how double integrals are really defined we need a two dimensional version of the pictures we had with many vertical strips (back in §10.2). What we do is think in terms of dividing up the region  $R$  (the floor in our picture) into a grid of rectangles (or squares). You could think of tiles on the floor, maybe better to think of very small tiles like mosaic tiles. We take the grid lines to be parallel to the  $x$  and  $y$  axes. Here is a picture of a grid, though this one has rather large rectangles.



One drawback we will see no matter how small we make the grid is that there will be irregular shapes at the edge. These do cause a bit of bother, but the idea is that if the grid is fine enough, the total effect of these incomplete rectangles around the edge should not matter much. Let's say we agree to omit any incomplete rectangles altogether.

Now concentrate on just one of the grid rectangles (or tiles on our floor), like the one marked in black in the picture. The part of the total volume under the roof that is above this one rectangle will make a sort of pillar, and it will be the analogue for the three dimensional picture of the graph  $z = f(x, y)$  of the rectangles we had in the §10.2 for functions of one variable.

You might look in the book to see nicely drawn artistic picture of what is going on here. We estimate the total volume under the 'roof'  $z = f(x, y)$  by adding up the volumes of the pillars we have just described. We square off the top of each pillar to make it a flat topped pillar of a height  $f(x, y)$  the same as the height of the graph at some point  $(x, y)$  inside our chosen grid rectangle (or floor tile). We get the volume of the pillar as the height  $f(x, y)$  times the area of the grid rectangle. If the side of the rectangle are  $\Delta x$  along the  $x$ -direction and  $\Delta y$  along the  $y$ -direction, that gives us a volume

$$f(x, y) \Delta x \Delta y$$

for that one pillar. We add all these up to get an estimate of the total volume 'under the graph'.

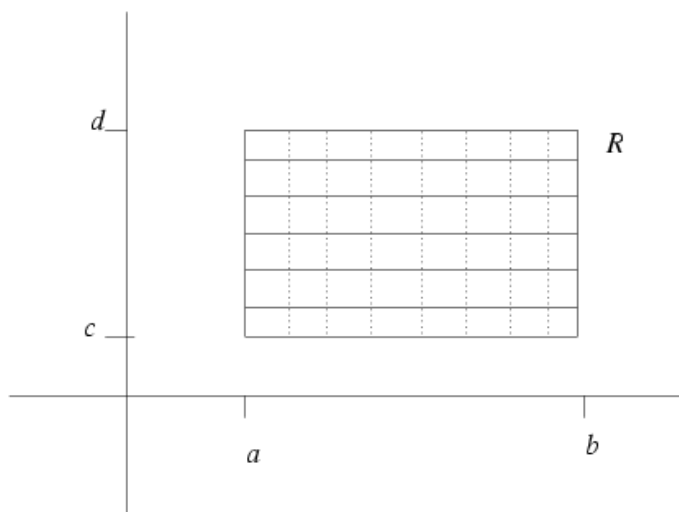
To get a better estimate, we should use more smaller grid rectangles. The integral is defined as the limit of the values we get by taking grids with smaller and smaller spacing between the grid lines.

### 3.1 Fubini's theorem

What we have so far is fairly theoretical or conceptual. It is in fact possible to make it practical with a computer program to calculate the sums we've been talking about, but for pencil and paper calculations we need something like the connection between definite integrals and antiderivatives we had for single integrals. The fact that works like that is called Fubini's theorem and we try to explain it first taking  $R$  to be a rectangle with sides parallel to the axes: say

$$R = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, c \leq y \leq d\}$$

(where  $a < b$  and  $c < d$  are numbers).



If we go back to thinking of the picture we had above, we now have a nice regular (rectangular) floor, though the roof above (the graph  $z = f(x, y)$ ) could be very irregular. Or maybe we should think of a cake (or loaf of bread) baked in a rectangular cake tin. We might need to be a bit imaginative about how the cake rose so as to allow for very strange shapes for the top of the cake according to a graph  $z = f(x, y)$ .

Now, you could explain our earlier strategy to estimate the volume of the solid object like this. First slice the cake into thin slices one way (say along the  $x$ -direction first) and then slice it in the perpendicular direction. We end up with rectangular ‘fingers’ of cake and we add up the volumes of each one. But we estimate the volume of the finger by treating it as a box with a fixed height, even though it will in fact have a slightly irregular top. (Not too irregular though if we make many cuts.)

To get some sort of explanation of why Fubini’s theorem works, we think of only slicing in one direction. Say we slice along the  $x$ -direction, perpendicular to the  $y$ -axis. That means  $y$  is constant along each slice. Think of cutting up the bread into slices, preferably very thin slices. (In our picture, if we cut in the  $x$ -direction we will be cutting down the length of the bread, rather than across. We would be allowed to cut the other way, keeping  $x$ -fixed, if we wanted, but let’s stick to the choice of cutting along the  $x$ -direction keeping  $y$ -fixed). We can think then of a thin slice and we could imagine finding its volume by taking the area of its face and multiplying by its thickness.

Say we call the thickness  $dy$  (for a very thin slice). How about the area of the face of the slice? Well we are looking at a profile of the slice as given by a graph  $z = f(x, y)$  with  $y$  fixed and  $x$  varying between  $a$  and  $b$ . So it has an area given by a single integral

$$\int_{x=a}^{x=b} f(x, y) dx$$

(where  $y$  is kept fixed, or treated as a constant, while we do this integral). We had some similar ideas when dealing with  $\partial f / \partial x$ . We kept  $y$  fixed and differentiated with respect to  $x$ . Or we took the graph of the function of one variable ( $x$  variable in fact) that arises by

chopping the graph  $z = f(x, y)$  in the  $x$ -direction, and looked at the slope of that. Anyhow, what we are doing now is integrating that function of  $x$  rather than differentiating it.

The integral above is the area of one slice. To get its volume we should multiply by the thickness  $dy$

$$\left( \int_{x=a}^{x=b} f(x, y) dx \right) dy$$

And then to get the total volume we should ‘add’ the volumes of the thin slices. Well, in the end we get an integral with respect to  $y$  and what Fubini’s theorem says is that

$$\iint_R f(x, y) dx dy = \int_{y=c}^{y=d} \left( \int_{x=a}^{x=b} f(x, y) dx \right) dy$$

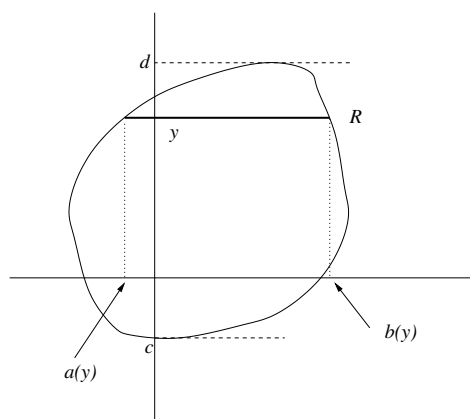
(when  $R$  is the rectangle  $R = [a, b] \times [c, d]$  we chose earlier).

Fubini’s theorem also allows us to do the integral with respect to  $y$  first (keeping  $x$  fixed) and then the integral with respect to  $x$ .

$$\iint_R f(x, y) dx dy = \int_{x=a}^{x=b} \left( \int_{y=c}^{y=d} f(x, y) dy \right) dx$$

In our explanation of why the theorem is true, this is what would happen if we sliced the other way. It may seem overkill to have two versions of the theorem, but there are examples where the calculations are much nicer if you do the  $dx$  integral first than if you do the  $dy$  integral first (and vice versa).

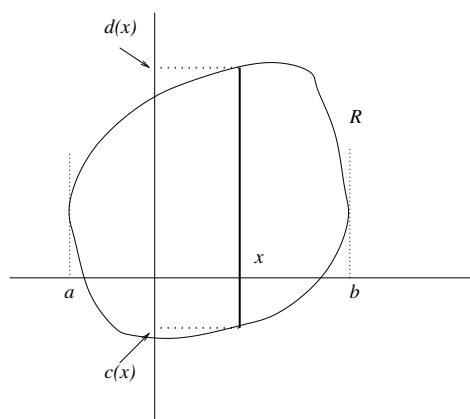
That was all for the case when  $R$  is a rectangle, but a similar idea will work even when  $R$  is more complicated. We can either integrate  $dy$  first and then  $dx$  or vice versa. But we need to take care that we cover all the points  $(x, y)$  in  $R$ . If we keep  $y$  constant at some value and then we have to figure out which  $x$ -values give  $(x, y)$  inside  $R$ . That range of  $x$ -values will now depend on which  $y$  we have fixed. In the picture that range of  $x$  values is  $a(y) \leq x \leq b(y)$ .



$$\iint_R f(x, y) \, dx \, dy = \int_{y=c}^{y=d} \left( \int_{x=a(y)}^{x=b(y)} f(x, y) \, dx \right) dy$$

The range  $c \leq y \leq d$  has to be so as to include all of  $R$  (and no more).

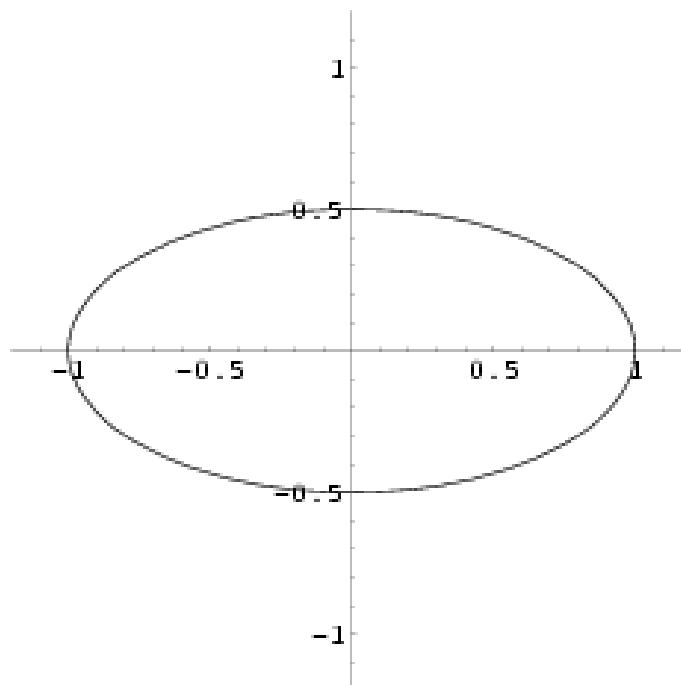
If we do the  $dy$  integral first, keeping  $x$  fixed, we need to figure out limits  $c(x) \leq y \leq d(x)$  for  $y$ . The idea should be clear from the following picture and the formula below.



$$\iint_R f(x, y) \, dx \, dy = \int_{x=a}^{x=b} \left( \int_{y=c(x)}^{y=d(x)} f(x, y) \, dy \right) dx$$

*Example 3.1.1.* Find  $\iint_R x^2 \, dx \, dy$  when  $R = \{(x, y) : x^2 + 4y^2 \leq 1\}$ .

It could help to draw  $R$  (it is the interior of an ellipse meeting the  $x$ -axis at  $(\pm 1, 0)$  and the  $y$ -axis at  $(0, \pm 1/2)$ ) but perhaps it is not really necessary to rely on the picture.



If we fix  $y$  and we want to know which  $x$ -values to integrate over (for that  $y$ ) it is not hard to figure it out:

$$\begin{aligned} x^2 + 4y^2 &\leq 1 \\ x^2 &\leq 1 - 4y^2 \\ -\sqrt{1 - 4y^2} &\leq x \leq \sqrt{1 - 4y^2} \end{aligned}$$

So the first integral to work out should be

$$\int_{x=-\sqrt{1-4y^2}}^{x=\sqrt{1-4y^2}} x^2 dx.$$

Then we have to integrate that over all  $y$  values for which there are any points. But it is fairly clear that there are points only when that square root is a square root of something that is positive. So we need

$$\begin{aligned} 1 - 4y^2 &\geq 0 \\ 1 &\geq 4y^2 \\ \frac{1}{4} &\geq y^2 \\ -\frac{1}{2} &\leq y \leq \frac{1}{2} \end{aligned}$$



We get

$$\begin{aligned}
 \iint_R x^2 dx dy &= \int_{y=-1/2}^{y=1/2} \left( \int_{x=-\sqrt{1-4y^2}}^{x=\sqrt{1-4y^2}} x^2 dx \right) dy \\
 &= \int_{y=-1/2}^{y=1/2} \left[ \frac{x^3}{3} \right]_{x=-\sqrt{1-4y^2}}^{x=\sqrt{1-4y^2}} dy \\
 &= \int_{y=-1/2}^{y=1/2} \frac{2(1-4y^2)\sqrt{1-4y^2}}{3} dy
 \end{aligned}$$

Let  $2y = \sin \theta$ ,  $2 dy = \cos \theta d\theta$ . For  $y = -1/2$ , we get  $-1 = \sin \theta$  and so  $\theta = -\pi/2$ . For  $y = 1/2$ , we get  $1 = \sin \theta$  and so  $\theta = \pi/2$ .

So, after the substitution, the integral we want becomes

$$\begin{aligned}
 &\int_{\theta=-\pi/2}^{\theta=\pi/2} \frac{2}{3}(1 - \sin^2 \theta) \sqrt{1 - \sin^2 \theta} \frac{1}{2} \cos \theta d\theta \\
 &= \int_{-\pi/2}^{\pi/2} \frac{1}{3} \cos^4 \theta d\theta \\
 &= \int_{-\pi/2}^{\pi/2} \frac{1}{3} \left( \frac{1}{2}(1 + \cos 2\theta) \right)^2 d\theta \\
 &= \frac{1}{12} \int_{-\pi/2}^{\pi/2} (1 + 2 \cos 2\theta + \cos^2 2\theta) d\theta \\
 &= \frac{1}{12} \int_{-\pi/2}^{\pi/2} \left( 1 + 2 \cos 2\theta + \frac{1}{2}(1 + \cos 4\theta) \right) d\theta \\
 &= \frac{1}{12} \int_{-\pi/2}^{\pi/2} \left( \frac{3}{2} + 2 \cos 2\theta + \frac{1}{2} \cos 4\theta \right) d\theta \\
 &= \frac{1}{12} \left[ \frac{3}{2}\theta + \sin 2\theta + \frac{1}{8} \sin 4\theta \right]_{-\pi/2}^{\pi/2} = \frac{\pi}{8}
 \end{aligned}$$

### 3.2 Applications of Double integrals

(i) If we take the case of the constant function 1,  $f(x, y) = 1$ ,

$$\begin{aligned}
 \iint_R f(x, y) dx dy &= \iint_R 1 dx dy \\
 &= \text{volume between the graph } z = 1 \text{ and} \\
 &\quad \text{the region } R \text{ in the } x\text{-}y \text{ (horizontal) plane} \\
 &= (\text{area of } R) \times (\text{height} = 1) \\
 &= \text{area}(R)
 \end{aligned}$$

In short, we have

$$\iint_R 1 \, dx \, dy = \text{area}(R)$$

- (ii) If  $R$  represents the shape of a thin (flat) plate, it will have a *surface density*  $\sigma(x, y)$ . By ‘surface density’ we mean mass per unit area and if the plate is the same thickness and the same material throughout that will be a constant

$$\sigma = \frac{\text{mass}}{\text{area}}.$$

But if it is not constant, we find the density at  $(x, y)$  by taking a very tiny sample of the plate at  $(x, y)$  and taking the mass per unit area of that. So if the sample is a rectangular piece of (infinitesimally short) sides  $dx$  in the  $x$ -direction times  $dy$  in the  $y$ -direction, then it has area  $dx \, dy$ . Say we call its mass  $dm$  (for a tiny bit of mass), then the surface density is

$$\sigma(x, y) = \frac{dm}{dx \, dy}$$

To get the total mass we should add up the little small masses  $dm$ , or actually integrate them. We get

$$\text{total Mass} = \iint_R dm = \iint_R \sigma(x, y) \, dx \, dy$$

## 4 Triple integrals

We now give a brief explanation of what a triple integral

$$\iiint_D f(x, y, z) \, dx \, dy \, dz$$

of a function  $f(x, y, z)$  of 3 variables over a region  $D$  in  $\mathbb{R}^3$  (space) means.

As we already know we cannot visualise the graph of a function of 3 variables in any realistic way (as it would require looking at 4 dimensional space) we cannot be guided by the same sort of graphical pictures as we had in the case of single integrals and double integrals.

We can proceed to think of sums, without the graphical back up. We think of  $D$  as a shape (solid shape) in space and we imagine chopping  $D$  up into a lot of small boxes with sides parallel to the axes. Think of slicing  $D$  first parallel to the  $x$ - $y$  plane, then parallel to the  $y$ - $z$  plane and finally parallel to the  $x$ - $z$  plane. Preferably we should use very thin separations between the places where we make the slices, but we have to balance that against the amount of work we have to do if we end up with many tiny boxes.

Look at one of the boxes we get from  $D$ . Say it has a corner at  $(x, y, z)$  and side lengths  $\Delta x$ ,  $\Delta y$  and  $\Delta z$ . We take the product

$$f(x, y, z) \Delta x \Delta y \Delta z = f(x, y, z) \times \text{Volume}.$$

We do this for each little box and add up the results to get a Riemann sum

$$\sum f(x, y, z) \Delta x \Delta y \Delta z$$

(We should agree some policy about what to do with the edges. If there are boxes that are not fully inside  $D$  we could agree to exclude them from the sum, say.) We now define the integral

$$\iiint_D f(x, y, z) dx dy dz = \lim (\text{these sums})$$

where we take the limit as the sums get finer and finer (in the sense that the maximum side length of the little boxes is becoming closer and closer to 0). If we are lucky, this limit makes sense — and there is a theorem that says the limit will make sense if  $D$  is bounded, not too badly behaved and  $f$  is a continuous function. So we will take it that the limit makes sense.

As for a picture to think about, we can think maybe of the example we used of mass. If  $f(x, y, z)$  is the *density* (= mass per unit volume) of a solid object occupying the region  $D$  of space, then

$$f(x, y, z) dx dy dz = \text{density} \times \text{volume} = \text{mass} = dm$$

(mass of a little tiny piece of the solid) and we get the total mass by adding these up. So

$$\iiint_D (\text{density function}) dx dy dz = \text{total mass}.$$

At least in the case where  $f(x, y, z) \geq 0$  always, we can interpret triple integrals this way.