From the fact that the level curves are circles we could immediately realise that the function is symmetrical around the origin, if we did not know that already. We can build the graph, which we saw before was a paraboloid, by lifting up the level curves $x^2 + y^2 = c$ to height c above the horizontal plane.

This may help you to see the relationship between the level curves and the graph. The map example, where they correspond to tracks you can walk along the landscape without going up or down at any stage, is also helpful.

Though a graph is maybe easier to visualise, it takes 3 dimensions. Maps with contour lines showing (say) the Mourne mountains could be harder to interpret, but they are easier to make than 3-dimensional models of the mountains.

2.4.10 Level curves and the gradient

Here we explain that the gradient vector at (x_0, y_0) of $f(x, y)$ will always be perpendicular to the level curve $f(x, y) = c$ that goes through the point (x_0, y_0). That is the one where $c = f(x_0, y_0)$.

Let us start with an example, the simple linear example $z = 1 + 2x - 3y$ and the point $(x_0, y_0) = (7, 11)$. We pictured the level curves (lines) for this earlier. So in this case we are dealing with the function $f(x, y) = 1 + 2x - 3y$ and we can work out the right value of c so that the level curve $f(x, y) = c$ passes through $(x_0, y_0) = (7, 11)$. We need $f(7, 11) = c$, or $1 + 14 - 33 = c$, $c = -18$. So the level curve is $1 + 2x - 3y = -18$ or $2x - 3y = -19$. As this is a line we can get its slope by rewriting it as $2x + 19 = 3y$, or $y = (2/3)x + 19/3$, so that the slope is $2/3$. If we find the gradient of f we get

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2, -3).$$

So the slope of the gradient vector is $-3/2$ and the product of this with the slope of the level curve (line) is $(-3/2)(2/3) = -1$. This shows they are perpendicular (in this example).

We take now another example $f(x, y) = x^2 + y^2$ and the point $(4, -1)$. We saw this example before and we know the level curves are concentric circles centered at the origin. So the tangent lines are perpendicular to the radius. In particular the tangent line to the circle through $(4, -1)$ is perpendicular to the vector $(4, -1)$. But if we work out the gradient vector we get

$$
\begin{aligned}
\nabla f \mid_{(4,-1)} &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \mid_{(4,-1)} \\
&= (2x, 2y) \mid_{(4,-1)} \\
&= (8, -2).
\end{aligned}
$$

As this gradient vector is a multiple of $(4, -1)$ we see that it is indeed perpendicular to the tangent line (to the level curve, or level circle in this example).

We've seen that the claimed fact that the gradient is always perpendicular to the tangent line to the level curve has worked out in two examples. Now we will indicate why it is always true.

The level curve $f(x, y) = c$ through a point (x_0, y_0) can be described in a different way by parametric equations

$$
\begin{cases}
x &= x(t) \\
y &= y(t)
\end{cases}
$$

in such a way that $(x(0), y(0)) = (x_0, y_0)$. We then have the equation

$$
f(x(t), y(t)) = c
$$

satisfied for all t. If we differentiate both sides with respect to t we get

$$
\begin{aligned}
\frac{d}{dt} f(x(t), y(t)) &= 0 \\
\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} &= 0
\end{aligned}
$$

by using the chain rule. When we put $t = 0$ this comes down to

$$
\left(\frac{\partial f}{\partial x} \mid_{(x_0, y_0)}, \frac{\partial f}{\partial y} \mid_{(x_0, y_0)} \right) \cdot \left(\frac{dx}{dt} \mid_{t=0}, \frac{dy}{dt} \mid_{t=0} \right) = 0.
$$

This dot product being zero means that the gradient vector of f at the point (x_0, y_0) is perpendicular to the velocity vector to the curve.

There are some details to fill in, but this is one way to prove that the gradient is perpendicular to the level curve.

Example 2.4.10.1. Find the equation of the tangent line to

$$
\cos(2x) \cos y = \frac{1}{2}
$$

at the point $(\pi/8, \pi/4)$. Also do the same at $(\pi/6, 0)$.

2.4.11 The implicit function theorem

It is a fact that that implicit differentiation is justifiable as long as it give a sensible answer (sensible meaning no division by zero). We can shed a little more light on that fact now, and perhaps explain a little better what is missing in the 'proof' above that the gradient vector is perpendicular to the level curve.

In implicit differentiation we assume that there was a function $y = y(x)$ that had a derivative and satisfied an equation in x and y. We could write that equation in x and y in terms of a function of two variables. Say the equation is

$$g(x, y) = 0$$

(when we put everything on the left of the equation and use the equation to define $g(x, y)$). Now we see that this relates to a level curve of $g(x, y)$, the level curve $g(x, y) = c$ where $c = 0$.

What we assumed then was that there was a differentiable function $y = y(x)$ so that

$$g(x, y(x)) = 0. \tag{1}$$

If we use the chain rule for partial derivatives we find that

$$\begin{aligned}
\frac{d}{dx}g(x, y(x)) &= \frac{\partial g}{\partial x}\frac{dx}{dx} + \frac{\partial g}{\partial y}\frac{dy}{dx} \\
&= \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{dy}{dx}
\end{aligned}$$

But, because of (1), we must get

$$\frac{d}{dx}g(x, y(x)) = 0$$

and so we have

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{dy}{dx} = 0.$$

Solving that gives

$$\frac{dy}{dx} = -\frac{\partial g/\partial x}{\partial g/\partial y},$$

which gives dy/dx in terms of x and y.

In implicit differentiation we have to assume we know y as well as x. That is we need a $y_0 = y(x_0)$ or one point (x_0, y_0) that satisfies $g(x, y) = 0$.

There is a theorem called the *implicit function theorem* that says:

If $g(x, y)$ is such that the partial derivatives $g_x(x, y)$ and $g_y(x, y)$ are continuous functions, and if we know both

$$g(x_0, y_0) = 0 \text{ and } g_y(x_0, y_0) \neq 0,$$

then there is a differentiable function $y = y(x)$ defined for x close to x_0 so that

$$g(x, y(x)) = 0 \qquad \text{(for all } x \text{ in the domain of } y(x)\text{)}.$$

Moreover the graph $y = y(x)$ fills the level curve $g(x, y) = 0$ near (x_0, y_0) and

$$\frac{dy}{dx} = y'(x) = -\frac{g_x(x, y(x))}{g_y(x, y(x))}.$$

This is quite tricky to prove, but it is the reason why implicitly defined functions are (usually) differentiable.

We could make a similar statement where the roles of x and y are reversed. So if $g(x_0, y_0) = 0$ and $g_x(x_0, y_0) \neq 0$, then there is a differentiable function $x = x(y)$ defined for y near y_0 that solves

$$g(x(y), y) = 0.$$

For that function we will get

$$\frac{dx}{dy} = -\frac{g_y(x(y), y)}{g_x(x(y), y)}.$$

Putting these two versions together, we see that if $\nabla g \mid_{(x_0,y_0)} \neq 0$ then we can solve the equation
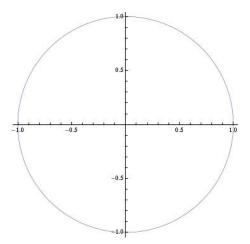
$$g(x, y) = 0$$

either for $y = y(x)$ as a differentiable function of x *or* for $x = x(y)$ as a differentiable function of y. We are not saying we can solve it everywhere, but just nearby the point we start with.

A very simple example comes from $g(x, y) = x^2 + y^2 - 1$. The level curve $g(x, y) = 0$ is the unit circle in this case and

$$\nabla g = (2x, 2y)$$

is never zero on the circle.