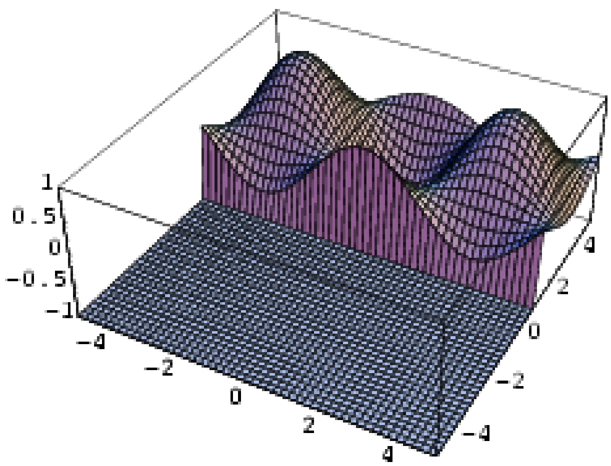
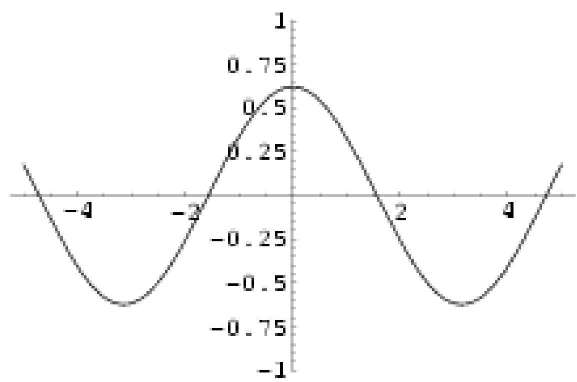
In terms of the graph $z = f(x, y)$ of the whole function of two variables, we can see the values $z = f(x, y_0)$ as those values along the line $y = y_0$ parallel to the $x$-axis in the (horizontal) $x$-$y$-plane. That means that the graph of the function of one variable $z = f(x, y_0)$ can be got from the whole graph of the function of two variables by taking a section at $y = y_0$.

Here is an attempt at a picture showing what the section of $z = \cos x \cos y$ at $y = 0.9$ looks like:




One is a three dimensional picture cutting away the part of the graph $z = f(x, y)$ where $y < 0.9$ and the other is a simple graph of the profile.

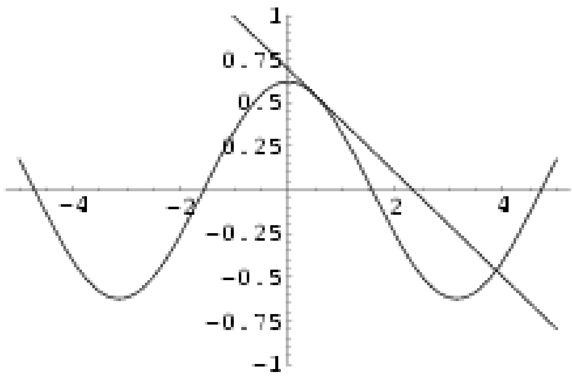
What we do now is take the derivative with respect to $x$ of this $f(x, y_0)$ and evaluate it at $x = x_0$. This gives the partial derivative at $(x_0, y_0)$, the partial derivative with respect to $x$. While we denote regular derivatives either by a $d/dx$ or a prime notation, we denote the partial derivative by $\partial/\partial x$. So the partial derivative evaluated at $(x_0, y_0)$ is denoted

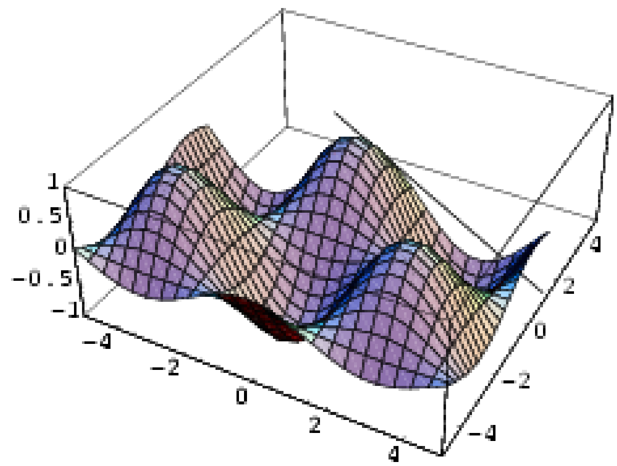
$$\frac{\partial f}{\partial x} |_{(x_0,y_0)}, \frac{\partial z}{\partial x} |_{(x_0,y_0)} \text{ , or sometimes } f_x(x_0, y_0).$$

(We will mostly use the $\partial f/\partial x$ notation rather than the $f_x$ notation, but the $f_x$ notation is also used quite often.) We could summarise the definition as

$$\frac{\partial f}{\partial x} |_{(x_0,y_0)} = \frac{d}{dx} |_{x=x_0} f(x, y_0)$$

If we want to see graphically what we are getting, recall that the ordinary derivative (of a function of one variable) is the slope of the tangent line to the graph at the point. So if for example, we were taking $f(x, y) = \cos x \cos y$, $(x_0, y_0) = (0.5, 0.9)$ we would be calculating the slope of this line




We can calculate this slope (or this derivative) exactly:

$$
\begin{aligned}
f(x, y_0) &= \cos x \cos y_0 \\
&= \cos x \cos 0.9 \\
\frac{d}{dx} f(x, y_0) &= -\sin x \sin y_0 \\
&= -\sin x \sin 0.9 \\
\frac{d}{dx} \,|_{x=x_0}\, f(x, y_0) &= -\sin x_0 \cos y_0 \\
&= -\sin 0.5 \cos 0.9 = -0.298016 \\
\frac{\partial f}{\partial x} \,|_{(x_0,y_0)} &= -0.298016
\end{aligned}
$$

Looking at the calculation, you can see that what we do is replace $y$ by the constant $y_0$ and then take the derivative $d/dx$ with respect to $x$ (and finally put $x = x_0$). We usually do it in a slightly quicker way. When taking the partial $\partial/\partial x$ with respect to $x$, we treat the variable $y$ as a constant. (We don't have to go so far as to replace $y$ by a specific number $y_0$.) If we repeat the above calculation this way, it would look like this:

$$
\begin{aligned}
f(x, y) &= \cos x \cos y \\
\frac{\partial f}{\partial x} &= -\sin x \cos y \\
\frac{\partial f}{\partial x} \,|_{(x_0,y_0)} &= -\sin x_0 \cos y_0 = -\sin 0.5 \cos 0.9 \\
&= -0.298016.
\end{aligned}
$$

**Summary:** *To get the partial derivative with respect to $x$, take the derivative with respect to $x$ while treating the variable $y$ as though it was a constant.*

Now partial derivatives with respect to $y$ are similar, except we freeze $x = x_0$ and differentiate with respect to a variable $y$ this time.

$$
\frac{\partial f}{\partial y} \,|_{(x_0,y_0)} = \frac{d}{dy} \,|_{y=y_0}\, f(x_0, y)
$$

Or, we can summarise by saying: *To get the partial derivative with respect to $y$, take the derivative with respect to $y$ treating the variable $x$ as though it was a constant.*

Graphically it means taking a section though our graph $z = f(x, y)$ in the $x$-direction, and then finding the slope of that.

If we take our example again $f(x, y) = \cos x \cos y$, $(x_0, y_0) = (0.5, 0.9)$, and find the partial with respect to $y$ this time, we find

$$\begin{aligned} f(x,y) &= \cos x \cos y \\ \frac{\partial f}{\partial y} &= \cos x(-\sin y) \\ &= -\cos x \sin y \\ \frac{\partial f}{\partial x}\,|_{(x_0,y_0)} &= -\cos x_0 \sin y_0 = -\cos 0.5 \sin 0.9 \\ &= -0.687434. \end{aligned}$$

*Remark* 2.4.2.1. It is necessary to become accustomed to taking partial derivatives. The rules are as for derivatives of functions of one variable, and so it is a question of getting practice in these again.

### 2.4.3 Directional derivatives

With partial derivatives we have concentrated on (vertical) sections in the directions parallel to the two axes (the $x$-axis for $\partial/\partial x$ and the $y$-axis for $\partial/\partial y$), but there seems to be no compelling reason to take only the slope of our graph $z = f(x, y)$ in these two directions.

Why should we not consider all possible directions through $(x_0, y_0)$? The slope we get is called a directional derivative.

To explain it, we should find a good way to describe a direction. We know vectors in the plane have a direction and a magnitude. As we are only interested in a direction, we can stick to unit vectors (magnitude = 1) $\mathbf{u}$ to describe a direction. Going through $(x_0, y_0)$ in the direction $\mathbf{u}$ we have a line and we know how to describe this line with parametric equations:

$$(x, y) = (x_0, y_0) + t\mathbf{u}$$

At $t = 0$ we have our point $(x_0, y_0)$ and we are heading in the direction specified by $\mathbf{u}$. If we now look at the values of $f(x, y)$ along this line, we get the function

$$t \mapsto f((x_0, y_0) + t\mathbf{u})$$

of the parameter (single variable) $t$. The graph of this function of $t$ is what we would get by taking a (vertical) section through the graph $z = f(x, y)$. It is the section through $(x_0, y_0)$ in the direction $\mathbf{u}$. We are going to take the slope of this section at the point $(x_0, y_0)$ we are concentrating on. We can see that the slope is the same[1] as the slope of the function

[1]For this to be true we are relying on the fact that $\mathbf{u}$ is a unit vector — if we allowed non unit vectors, the scale along the $t$ axis would not correspond to the distance scale in the horizontal $(x, y)$ plane, and this would affect the calculation of the slope.

of $t$, that is the same as the derivative with respect to $t$ at $t = 0$:

$$\frac{d}{dt}\Big|_{t=0} f((x_0, y_0) + t\mathbf{u})$$

We *define* the directional derivative $D_{\mathbf{u}}f(x_0, y_0)$ of $f$ in the direction $\mathbf{u}$ at $(x_0, y_0)$ to be

$$D_{\mathbf{u}}f(x_0, y_0) = \frac{d}{dt}\Big|_{t=0} f((x_0, y_0) + t\mathbf{u})$$

We can notice that if we take $\mathbf{u} = \mathbf{i} = (1, 0) =$ the unit vector in the direction of the positive $x$-axis, then

$$\begin{aligned}
D_{\mathbf{u}}f(x_0, y_0) &= D_{\mathbf{i}}f(x_0, y_0) \\
&= \frac{d}{dt}\Big|_{t=0} f((x_0, y_0) + t\mathbf{i}) \\
&= \frac{d}{dt}\Big|_{t=0} f((x_0 + t, y_0)) \\
&= \frac{d}{dx}\Big|_{x=x_0} f(x, y_0) \\
&= \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}
\end{aligned}$$

Similarly for $\mathbf{u} = \mathbf{j} = (0, 1)$ the directional derivative $D_{\mathbf{j}}f$ is the same as $\partial f/\partial y$.

We can compute what $D_{\mathbf{u}}f(x_0, y_0)$ for the example we have used before, that is $f(x, y) = \cos x \cos y$ and $(x_0, y_0) = (0.5, 0.9)$. We'll work it out for an arbitrary unit vector $\mathbf{u} = (u_1, u_2)$.

$$\begin{aligned}
D_{\mathbf{u}}f(x_0, y_0) &= \frac{d}{dt}\Big|_{t=0} f((x_0, y_0) + t\mathbf{u}) \\
&= \frac{d}{dt}\Big|_{t=0} f(x_0 + tu_1, y_0 + tu_2) \\
&= \frac{d}{dt}\Big|_{t=0} \cos(x_0 + tu_1)\cos(y_0 + tu_2) \\
&= (-u_1 \sin(x_0 + tu_1)\cos(y_0 + tu_2) - \cos(x_0 + tu_1)u_2 \sin(y_0 + tu_2))\Big|_{t=0} \\
&= -u_1 \sin x_0 \cos y_0 - u_2 \cos x_0 \sin y_0 \\
&= -\sin x_0 \cos y_0 u_1 - \cos x_0 \sin y_0 u_2 \\
&= -0.298016u_1 - 0.687434u_2.
\end{aligned}$$

You'll notice that the same numbers are coming in as we had for the partial derivatives. Of course this was just one example of a very simple kind, and the fact that the directional derivative involves the same numbers we got from the partial derivatives might be a fluke.