‘

Theorem 1.4.3 (continued from previous page)**.**

(c) Moreover,if $R > 0$, the series $\sum_{n=1}^{\infty} na_n(x-a)^{n-1}$ has the same radius of convergence and so $f'(x)$ is differentiable (for $|x-a| < R$).

(d) In fact, assuming $R > 0$, $f(x)$ can be differentiated any number of times and the coefficients a_n are related to the values of the derivatives at a by

$$a_n = \frac{1}{n!} f^{(n)}(a).$$

(e) We can also integrate 'term by term' rather than differentiate and say that

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$$

has the same radius if convergence R and is an antiderivative for $f(x)$ (when $|x-a| < R$).

Example 1.4.4. The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

has radius of convergence $R = \infty$.

To prove that we can use the ratio test (Theorem 1.3.21) to show that the series converges for each x.

It might be no harm to recall that $n!$ (n factorial) means the product of the numbers $1, 2, \ldots, n$ but we also define $0! = 1$. Recall also that $x^0 = 1$ and so our series becomes

$$1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \tag{1}$$

We stated the ratio test for series indexed from 1 (rather than from 0) but that makes no essential difference. If we wanted to be fussy we could renumber all the terms of the power series $t_1 = 1$, $t_2 = x$, $t_3 = x^2/2$ and so on.

For the ratio test we take the limit of the absolute value of the ratios of each term to the term before

$$\lim_{n\to\infty} \frac{\left|\frac{1}{(n+1)!}x^{n+1}\right|}{\left|\frac{1}{n!}x^n\right|} = \lim_{n\to\infty} \frac{n!|x|^{n+1}}{(n+1)!|x|^n} = \lim_{n\to\infty} \frac{|x|}{(n+1)} = 0$$

Therefore, by the ratio test, the series converges for all x and that tells us $R = \infty$ is the radius of convergence.

It follows then that we can define a function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

and then (according to Theorem 1.4.3) we can calculate $f'(x)$ by differentiating the series 'term by term'. It may be easier to see what happens if we look at the series for $f(x)$ in the form (1). We get

$$f'(x) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots = 1 + x + \frac{x^2}{2!} + \cdots = f(x)$$

Do we know any function which is its own derivative? Well yes, the exponential function e^x. In fact we can show via a useful trick that this $f(x)$ is e^x.

Lemma 1.4.5. *If $y = f(x)$ is any function that satisfies the differential equation*

$$\frac{dy}{dx} - y = 0$$

then

$$y = Ce^x$$

for some C (constant).

Proof. There is a trick for this called the *method of integrating factors*. We multiply both sides of the equation

$$\frac{dy}{dx} - y = 0$$

by e^{-x} and we get

$$e^{-x}\frac{dy}{dx} - e^{-x}y = 0,$$

which we can rewrite

$$e^{-x}\frac{dy}{dx} + \left(\frac{d}{dx}(e^{-x})\right) y = 0,$$

and then recognise the left hand side as the derivative of a product

$$\frac{d}{dx}(e^{-x}y) = 0.$$

But then $e^{-x}y$ has to be a constant C, that is

$$e^{-x}y = C.$$

Multiply across by e^x to get $y = Ce^x$ $\square$

Corollary 1.4.6.

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

Proof. We saw that $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ satisfies $f'(x) = f(x)$ (or, in other words, that $y = f(x)$ solves the differential equation $dy/dx = y$ or $dy/dx - y = 0$. So by the Lemma (or the integrating factor trick) we must have $y = f(x) = Ce^x$. But for $x = 0$ we can see that $f(0) = 1 + 0 + 0 + \cdots = 1$ and we also have $f(0) = Ce^0 = C$. So we must have $C = 1$. $\square$

Remark 1.4.7. The integrating factor trick can be used to solve linear constant coefficient differential equations of the first order, that is ones of the form

$$\frac{dy}{dx} + ky = g(x)$$

for some constant k and some right hand side $g(x)$ depending only on x.

[A *differential equation* is an equation where the unknown is a function $y = f(x)$ and the equation involves not only x and y but derivatives of the unknown function y. *First oder* means only the first derivative. *Linear* means the equation is linear in y.]

The trick is to multiply by e^{kx} and then the left hand side becomes the derivative of a product. We get

$$\frac{d}{dx}(e^{kx}y) = g(x)e^{kx}$$

Provided we can find an antiderivative (indefinite integral) $h(x)$ for $g(x)e^{kx}$, that is a function $h(x)$ with $h'(x) = g(x)e^{kx}$, we now have

$$\frac{d}{dx}(e^{kx}y) = h'(x)$$

or

$$\frac{d}{dx}(e^{kx}y - h(x)) = 0$$

and so $e^{kx}y - h(x) = C =$ a constant. Rearrange this to get y (up to not knowing C — we need another bit of information to find C).

Example 1.4.8. We know the sum of a geometric series

$$1 + x + x^2 + \cdots = \frac{1}{1-x}$$

for $|x| < 1$, while the series does not converge for $|x| > 1$. [Why not? Well the terms x^n do not tend to 0 as $n \to \infty$ and so there is no chance that the series can converge if $|x| > 1$. That's what Theorem 1.3.4 says.]

So this a a power series (centered at $a = 0$) with radius of convergence $R = 1$. For $|x| < 1$ we must have the derivative

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = 0 + 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} nx^n$$

or

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n$$

Instead of differentiating we could also integrate the series. We get

$$\frac{d}{dx}\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) = \frac{d}{dx}\sum_{n=0}^{\infty} \frac{1}{n+1}x^{n+1} = \sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

(for $|x| < 1$) and so

$$\sum_{n=1}^{\infty} \frac{1}{n}x^n = -\ln(1-x) = \ln\left(\frac{1}{1-x}\right)$$

In this way it is possible to arrive at formulae for sums of series. For example take $x = 1/2$ in the last formula. We get $\sum_{n=1}^{\infty} 1/(n2^n) = \ln 2$.

Remark 1.4.9. If we knew about the exponential function $f(x) = e^x$ and we were hoping to find a series (centered at $a = 0$ say) that summed to it, we know we would have to take the coefficients to be

$$a_n = \frac{f^{(n)}(a)}{n!} = \frac{e^a}{n!}$$

(Look at Theorem 1.4.3 (d)). In the case of the exponential function, it turns out that $\sum_{n=0}^{\infty} a_n(x-a)^n = e^x$ works for all x, but that does not happen for every function. Sometimes it works, sometimes for all x, sometimes with a finite radius of convergence and sometimes it just does not work expect for $x = a$.

There is a theorem called Taylor's theorem which can sometimes be used to prove that $f(x)$ is given by a power series.

Definition 1.4.10. If $f(x)$ is n-times differentiable at a, then the *Taylor polynomial of degree n for f centered at a* is

$$P_{n,f,a}(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(Note that they would be the partial sums of the power series, if there was a power series that gave $f(x)$ near $x = a$.)

Taylor's Theorem gives a formula for $f(x) - P_{n,f,a}(x)$ (which sometimes can be used to show $\lim_{n\to\infty} P_{n,f,a}(x) = f(x)$). We won't explain that in any more detail.