Theorem 1.3.18 (Comparison test — improved version)**.** *Suppose* $\sum_{n=1}^{\infty} x_n$ *and* $\sum_{n=1}^{\infty} y_n$ *are two series and suppose* $|x_n| \leq y_n$ *for each* n*. Then*

(a) if the larger (nonnegative) series $\sum_{n=1}^{\infty} y_n$ *converges then the smaller series* $\sum_{n=1}^{\infty} x_n$ *converges (absolutely);*

(b) if the smaller series $\sum_{n=1}^{\infty} x_n$ *does not converge then neither does the larger one* $\sum_{n=1}^{\infty} y_n$*.*

Proof. This really is just a combination of the previous version of the comparison test (for series of nonnegative terms) with remarks above about absolute convergence.

If $\sum_{n=1}^{\infty} y_n$ converges then $\sum_{n=1}^{\infty} |x_n|$ converges by the previous version of this test (Theorem 1.3.8 (a)). But that means exactly that $\sum_{n=1}^{\infty} x_n$ is absolutely convergent, hence convergent by Theorem 1.3.16.

If $\sum_{n=1}^{\infty} x_n$ is not convergent, then it can't be absolutely convergent (because of Theorem 1.3.16), which says that $\sum_{n=1}^{\infty} |x_n|$ does not converge. Then $\sum_{n=1}^{\infty} y_n$ can't converge because of previous version of this test (Theorem 1.3.8 (b)). □

To exhibit examples of series that are convergent but not absolutely convergent there is the following test.

Theorem 1.3.19 (Alternating series test)**.** *Suppose* $\sum_{n=1}^{\infty} x_n$ *is a series where the terms alternate in sign (so that* $x_{n+1}/x_n < 0$ *always). Assume also that the absolute values of the terms are decreasing to 0, that is*

$$|x_1| \geq |x_2| \geq |x_3| \geq \cdots \geq |x_n| \geq |x_{n+1}| \geq \cdots \text{ and } \lim_{n\to\infty} x_n = 0.$$

Then the series is convergent.

Proof. We'll prove it for the case where the first term $x_1 > 0$. The case where $x_1 < 0$ would be similar or we could change to the first case by multiplying by -1 (so that $\sum_{n=1}^{\infty}(-x_n)$ converges by the first case and then we can multiply by -1 again to get that $\sum_{n=1}^{\infty} x_n$ converges).

Consider the partial sums $s_n = \sum_{j=1}^{n} x_j$ and focus first on the even partial sums. We can group the terms in pairs so that

$$s_{2n} = (x_1 + x_2) + (x_3 + x_4) + \cdots (x_{2n-1} + x_{2n}) = \sum_{j=1}^{n} (x_{2j-1} + x_{2j})$$

But all these bracketed sums of two successive terms are positive (or at least nonnegative) because $x_{2j-1} > 0$, $x_{2j} < 0$ and

$$x_{2j-1} + x_{2j} = |x_{2j-1}| - |x_{2j}| \geq 0.$$

Therefore the sequence of even partial sums $(s_{2n})_{n=1}^\infty$ is monotone increasing. If we can show that it is bounded above then the even partial sums will have a limit and we will be nearly done.

If we look at the odd partial sums in a similar way we see that they can be written

$$s_{2n-1} = x_1 + (x_2 + x_3) + (x_4 + x_5) + \cdots + (x_{2n-1} + x_{2n-1})$$

and the bracketed sums of two successive terms are now all negative (or not positive because $x_{2j-2} + x_{2j-1} = -|x_{2j-1}| + |x_{2j}| \leq 0$). So they are decreasing $x_1 = s_1 \geq s_3 \geq s_5 \geq \cdots$.

Now

$$s_{2n} = s_{2n-1} + x_{2n} \leq s_{2n-1} \leq s_1 = x_1$$

and so the even partial sums are bounded above (by the first term x_1). So

$$\lim_{n\to\infty} s_{2n}$$

exists (as a finite limit). We need to know that the odd partial sums have the same limit, but that is not so hard to see because

$$s_{2n+1} = s_{2n} + x_{2n+1}$$

and $\lim_{n\to\infty} x_{2n+1} = 0$. $\square$

Example 1.3.20. For $p > 0$, the series

$$\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \frac{1}{5^p} - \cdots$$

converges (by the alternating series test) but it is not absolutely convergent if $0 < p \leq 1$.

Proof. The ratio of successive terms is negative, the absolute values $|(-1)^{n-1}/n^p| = 1/n^p$ are decreasing with n and $\lim_{n\to\infty} 1/n^p = 0$.

We know the series of absolute values $\sum_{n=1}^\infty 1/n^p$ fails to converge for $0 < p \leq 1$ (see Examples 1.3.9 (iii) where $p \leq 0$ is included also — but the alternating series test does not apply to $\sum_{n=1}^\infty (-1)^{n-1}/n^p$ for $p \leq 0$ because the terms don't decrease in absolute value). $\square$

Theorem 1.3.21 (Ratio test)**.** *Suppose $\sum_{n=1}^\infty x_n$ is a series and*

$$r = \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}$$

exists (as a finite nonnegative limit or as ∞).

(i) if $r < 1$ then the series is (absolutely) convergent;

(ii) if $r > 1$ then the series does not converge;

(iii) if $r = 1$ no conclusion can be drawn about convergence of the series from this information alone.

Note that the limit we are considering is the limit of the ratio of a term of the series to the previous one. To be more precise the absolute value of that ratio.

Proof. Suppose $r < 1$ first. Put $\rho = (1 + r)/2$ so that $r < \rho < 1$. Since the limit of $|x_{n+1}|/|x_n|$ is r and $r < \rho$, eventually we must have $|x_{n+1}|/|x_n| < \rho$.

In more precise terms (by using $\varepsilon = \rho - r$ in the ε-N definition of the limit), there is N so that

$$n > N \Rightarrow \frac{|x_{n+1}|}{|x_n|} < \rho$$

The idea is to compare the series $\sum_{n=1}^{\infty} x_n$ to a suitable multiple of the geometric series $\sum_{n=1}^{\infty} \rho^n$.

We take the mutltiple big enough to deal with all terms up to x_{N+1}, that is we take

$$k = \max_{1 \le j \le N+1} \frac{|x_j|}{\rho^{j-1}}.$$

So then we have $|x_j| \le k\rho^{j-1}$ for $j = 1, 2, \ldots, N + 1$. We can then deduce that

$$\frac{|x_{N+2}|}{|x_{N+1}|} < \rho \Rightarrow |x_{N+2}| < \rho|x_{N+1}| \le \rho k \rho^{N+1-1} = k\rho^{(N+2)-1}$$

and show by induction that

$$|x_{N+j}| \le k\rho^{(N_j)-1} \text{ for all } j \in \mathbb{N}.$$

Combining that with the way we picked k we have

$$|x_n| \le k\rho^{n-1}$$

for all n. Since $\sum_{n=1}^{\infty} k\rho^{n-1}$ converges (it is $k\sum_{n=1}^{\infty} \rho^{n-1} = k/(1-\rho)$), the comparison test tells us that $\sum_{n=1}^{\infty} x_n$ converges (in fact converges absolutely). □

Example 1.3.22. For which values of x does $\sum_{n=1}^{\infty} \sqrt{n}x^n$ converge?
Solution: Although x is a variable, for fixed x we have a series $\sum_{n=1}^{\infty} t_n$ where the n^{th}-term is

$$t_n = \sqrt{n}x^n$$

To apply the ratio test, we should evaluate

$$r = \lim_{n\to\infty} \frac{|t_{n+1}|}{|t_n|} = \lim_{n\to\infty} \frac{|\sqrt{n+1}x^{n+1}|}{|\sqrt{n}x^n|} = \lim_{n\to\infty} \sqrt{1 + 1/n}|x| = |x|$$

Since there is a limit we can say that the series definitely converges for $|x| < 1$ (that is for $-1 < x < 1$) and does not converge for $|x| > 1$. The uncertain values of x are just $x = \pm 1$.

In many problems of this type, it can be difficult to deal with these remaining cases. However, in this example the series for $x = 1$ is $\sum_{n=1}^{\infty} \sqrt{n}$ and for $x = -1$ it is $\sum_{n=1}^{\infty} (-1)^n\sqrt{n}$. Neither of these can converge because the terms do not tend to 0 as $n \to \infty$.

1.4 Power series

Definition 1.4.1. A *power series* is a series of the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n$$

where x is a variable, a is a fixed point called the centre of the series and the numbers a_0, a_1, ... are called the coefficients of the series.

In series of this type it is usual to have a convention that $(x-a)^0$ means 1, even for $x = a$ (where it would read 0^0, something that one should not normally attempt to define).

So the power series means

$$a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

When $a = 0$ the series looks like

$$a_0 + a_1x + a_2x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

Example 1.4.2. In the previous example we have $a = 0$ and $a_n = \sqrt{n}$.

We don't have any simple formula for the sum of that series. From the formula for the sum of a geometric series we know

$$1 + (x-a) + (x-a)^2 + \cdots = \sum_{n=0}^{\infty}(x-a)^n = \sum_{n=1}^{\infty}(x-a)^{n-1} = \frac{1}{1-(x-a)}$$

for $|x-a| < 1$. (Here the coefficient of $(x-a)^n$ is $a_n = 1$ for each $n \geq 0$.)

We now summarise (without any attempt at a proof) some of the main facts about power series.

Theorem 1.4.3. *Let $\sum_{n=1}^{\infty} a_n(x-a)^n$ be a power series.*

(a) The series has a radius of convergence, *a 'number' $R \geq 0$ with the property that the series converges for $|x-a| < R$ but fails to converge for any x with $|x-a| > R$.*

However the 'number' R may be ∞, that is it may be that the power series converges for all x. (It may also be $R = 0$, in which case the series converges only for $x = a$.)

(b) If $R > 0$ we can define $f\colon (a-R, a+R) \to \mathbb{R}$ by

$$f(x) = \sum_{n=1}^{\infty} a_n(x-a)^n$$

The function $f(x)$ will be diffentiable and its derivative is given by

$$f'(x) = \sum_{n=1}^{\infty} n a_n(x-a)^{n-1}$$

for $|x-a| < R$.