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Examples 1.3.9. (i) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Proof. By use of telescoping sums we saw in Examples 1.3.3 that $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges.

Now
$$\frac{1}{n^2+n} \geq \frac{1}{2n^2} \quad (\text{for all } n \geq 1)$$
[because $n \leq n^2$] and so by the comparison test
$$\sum_{n=1}^{\infty} \frac{1}{2n^2}$$
must converge (smaller series of positive terms). Multiplying by 2 we see that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ must converge also (see question 2 on tutorial sheet 2). □

(ii) For any $p \geq 2$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

Proof. Since for $p \geq 2$
$$\frac{1}{n^p} \leq \frac{1}{n^2}$$
the comparison test (together with the previous example) shows that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. □

(iii) For any $p \leq 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ does not converge.

Proof. In this case we have $n^p \leq n$ and so
$$\frac{1}{n^p} \geq \frac{1}{n} \quad (\text{for all } n \geq 1)$$
Therfore $\sum_{n=1}^{\infty} 1/n^p$ cannot converge by the comparison test — because the smaller harmonic series $\sum_{n=1}^{\infty} 1/n$ does not converge (Example 1.3.5). □

Notice that we have not dealt with $\sum_{n=1}^{\infty} 1/n^p$ for $1 < p < 2$. There is a way to settle that with integrals, using the following result.

Theorem 1.3.10 (Integral test)**.** *Suppose $f\colon [1,\infty) \to [0,\infty)$ is a decreasing continuous function. Then the series*

$$\sum_{n=1}^{\infty} f(n) \text{ converges}$$

if and only if there is a finite number u so that

$$\int_1^b f(x)\,dx \leq u \text{ holds for all } b \geq 1$$

So the condition means that there is a finite upper bound for the integrals $\int_1^b f(x)\,dx$ $(b > 1)$.

(Note that since $f(x) \geq 0$ the integrals $\int_1^b f(x)\,dx$ increase when b increases. If these numbers are bounded above they have a least upper bound which coincides with $\lim_{b\to\infty} \int_1^b f(x)\,dx$. The test is often phrased in terms of this limit being finite.)

Proof. The proof is based on the fact that

$$f(n) \geq f(x) \geq f(n+1)$$

for $n \leq x \leq n+1$ (because $f(x)$ is decreasing with x) and so

$$\int_n^{n+1} f(n)\,dx \geq \int_n^{n+1} f(x)\,dx \geq \int_n^{n+1} f(n+1)\,dx$$

which means

$$f(n) \geq \int_n^{n+1} f(x)\,dx \geq f(n+1)$$

It follows that

$$\sum_{j=1}^{n} f(j) \geq \sum_{j=1}^{n} \int_j^{j+1} f(x)\,dx \geq \sum_{j=1}^{n} f(j+1)$$

and so

$$\sum_{j=1}^{n} f(j) \geq \int_1^{n+1} f(x)\,dx \geq \sum_{j=2}^{n+1} f(j) = \left(\sum_{j=1}^{n+1} f(j)\right) - f(1)$$

So if the series $\sum_{n=1}^{\infty} f(n)$ converges, there is a finite upper bound s for its partial sums and we have

$$\int_1^{n+1} f(x)\,dx \leq \sum_{j=1}^{n} f(j) \leq s$$

for all n. For any $b \geq 1$ we can find n with $b \leq n+1$ and then

$$\int_1^{b} f(x)\,dx \leq \int_1^{n+1} f(x)\,dx \leq \sum_{j=1}^{n} f(j) \leq s$$

So the integrals must be bounded above if the series converges.

On the other hand if there is $u \in \mathbb{R}$ with $\int_1^b f(x)\,dx \leq u$ for all $b \geq 1$, then we have

$$\sum_{j=1}^{n+1} f(j) \leq f(1) + \int_1^{n+1} f(x)\,dx \leq f(1) + u$$

The partial sums of the series $\sum_{n=1}^\infty f(n)$ are bounded above (by $f(1)+u$) and so the series converges (Theorem 1.3.7). □

Example 1.3.11. $\sum_{n=1}^\infty 1/n^p$ converges for $p > 1$.

Proof. Take $f(x) = 1/x^p$ in the integral test.

$$\int_1^b \frac{1}{x^p}\,dx = \int_1^b x^{-p}\,dx = \left[\frac{x^{1-p}}{1-p}\right]_1^b = \frac{b^{1-p}}{1-p} - \frac{1}{1-p} = \frac{1}{p-1} - \frac{b^{p-1}}{p-1} \leq \frac{1}{p-1}$$

Since the integrals are bounded above, and $f(x)$ is decreasing and positive for $x \geq 1$, the integral test tells us that the series converges. □

By the way we could use the integral test to give new proofs that the harmonic series $\sum_{n=1}^\infty 1/n$ does not converge and more generally that $\sum_{n=1}^\infty 1/n^p$ does not converge for p in the range $0 < p \leq 1$.

Proposition 1.3.12. *If $\sum_{n=1}^\infty x_n$ and $\sum_{n=1}^\infty y_n$ are two convergent series, then $\sum_{n=1}^\infty (x_n + y_n)$ is also convergent and has sum*

$$\sum_{n=1}^\infty (x_n + y_n) = \sum_{n=1}^\infty x_n + \sum_{n=1}^\infty y_n$$

Proof. This is really simple using the fact that the sum of a series means the limit of the partial sums, plus the fact that the n^{th} partial sum of the series $\sum_{n=1}^\infty (x_n + y_n)$ is

$$\begin{aligned}
\sum_{j=1}^n (x_j + y_j) &= (x_1 + y_1) + (x_2 + y_2) + \cdots + (x_n + y_n) \\
&= (x_1 + x_2 + \cdots + x_n) + (y_1 + y_2 + \cdots + y_n) \\
&= \sum_{j=1}^n x_j + \sum_{j=1}^n y_j
\end{aligned}$$

The limit as $n \to \infty$ of this is $\sum_{n=1}^\infty x_n + \sum_{n=1}^\infty y_n$ according to the theorem on limits of sums of sequences (Theorem 1.1.6 (iii)). □

Remark 1.3.13. An equally simple fact that if $\sum_{n=1}^\infty x_n$ converges and k is a constant, then

$$\sum_{n=1}^\infty (kx_n) = k \sum_{n=1}^\infty x_n$$

was on Tutorial 2 (and we used it above in Examples 1.3.9 (i)).

Notation 1.3.14. *For a real number x we write*

$$x_+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and

$$x_- = \begin{cases} 0 & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

We call x_+ the positive part *of x and x_- the* negative part *— though both parts are actually positive.*

Here are the key facts about them (true for each $x \in \mathbb{R}$):

$$x = x_+ - x_-, \quad x_+ \geq 0, \quad x_- \geq 0, \quad x_+ x_- = 0,$$
$$|x| = x_+ + x_-, \quad x_+ = \frac{|x| + x}{2}, \quad x_- = \frac{|x| - x}{2}.$$

Definition 1.3.15. A series $\sum_{n=1}^{\infty} x_n$ is called *absolutely convergent* if the series $\sum_{n=1}^{\infty} |x_n|$ is convergent.

Theorem 1.3.16. *Absolutely convergent series are convergent.*

That is if $\sum_{n=1}^{\infty} |x_n|$ converges, then so does $\sum_{n=1}^{\infty} x_n$.

Proof. Suppose $\sum_{n=1}^{\infty} |x_n|$ converges. Then notice that

$$0 \leq (x_n)_+ \leq |x_n|$$

So, by comparison $\sum_{n=1}^{\infty} (x_n)_+$ must converge. Similarly (because $0 \leq (x_n)_- \leq |x_n|$) $\sum_{n=1}^{\infty} (x_n)_-$ must converge too.

Multiplying that by -1 we get

$$\sum_{n=1}^{\infty} -(x_n)_-$$

converges. Adding that series to $\sum_{n=1}^{\infty} (x_n)_+$ we get that

$$\sum_{n=1}^{\infty} ((x_n)_+ - (x_n)_-) = \sum_{n=1}^{\infty} x_n$$

converges. □

Example 1.3.17. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges.

Proof. Since $|\cos x| \leq 1$ always,

$$\left| \frac{\cos n}{n^2} \right| = \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$$

Since we know $\sum_{n=1}^{\infty} 1/n^2$ converges, the comparison test tell us that $\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$ converges.

So now we know $\sum_{n=1}^{\infty} (\cos n)/n^2$ is absolutely convergent, therefore convergent. □