‘

If the series $\sum_{n=1}^{\infty} x_n$ is convergent then the limit of the partial sums, $\lim_{n\to\infty} \sum_{j=1}^{n} x_j$ is defined to be the *sum* of the series.

In a notation that is totally standard but also potentially confusing, $\sum_{n=1}^{\infty} x_n$ is used for two things. One is to say that we are discussing a series, wondering then whether or not it converges, and the other is to represent the sum of the series when it does converge:

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} \sum_{j=1}^{n} x_j$$

Another term used for convergent series is *summable series.*

Examples 1.3.3. There are a small number of series where we can show that they are convergent and also compute their sums by direct and elementary methods.

(i) (Telescoping sums)

Suppose the series $\sum_{n=1}^{\infty} x_n$ has n^{th} term

$$x_n = \frac{1}{n} - \frac{1}{n+1}$$

which could also be rewritten

$$x_n = \frac{1}{n} - \frac{1}{n+1} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n^2+n}$$

Then the partial sums are

$$s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

So the sum of the series is

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1.$$

(ii) (not such a good example)

If $x_n = 1$ for each n, then $s_n = \sum_{j=1}^{n} 1 = n$ and so $\lim_{n\to\infty} s_n$ is infinite. The series

$$\sum_{n=1}^{\infty} 1$$

does *not* converge.

(iii) (Geometric series)

Suppose there there is a number r with $|r| < 1$ and $x_n = r^{n-1}$. The sequence $1, r, r^2, \ldots$ is called a *Geometric Progression* (or GP) and the corresponding series is called a *Geometric series*. At least if $r \neq 0$ the ratio of successive terms is

$$\frac{x_{n+1}}{x_n} = \frac{r^{n+1}}{r^n} = r$$

(constant ratio r, also known as a common ratio).

There is a formula for the partial sums

$$s_n = 1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}$$

There are various ways to prove this, for instance by induction on n, but a simple proof is to write down

$$rs_n = r + r^2 + r^3 + \cdots + r^n$$

and subtract that from s_n (cancelling all the common terms in the sums) to get

$$s_n - rs_n = 1 - r^n$$

That gives $(1 - r)s_n = 1 - r^n$ and then we can divide by $1 - r$ to get the formula. [By the way that formula is good for $r \neq 1$, not just for $|r| < 1$.]

Now

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} r^{n-1} = \lim_{n\to\infty} \sum_{j=1}^{n} r^{j-1} = \lim_{n\to\infty} \frac{1 - r^n}{1 - r} = \frac{1}{1 - r}$$

Here we are relying on $\lim_{n\to\infty} r^n = 0$ for $|r| < 1$, which we will not prove.

The main point of the next few results is to get an understanding of which series are convergent (or summable). However the short answer is that it is a rather subtle question, with no simple method to decide about convergence — at least no method that always works.

Theorem 1.3.4. *(Terms must tend to 0)*
If $\sum_{n=1}^{\infty} x_n$ is a convergent series, then

$$\lim_{n\to\infty} x_n = 0$$

Proof. Write s_n as usual for the n^{th} partial sum $s_n = \sum_{j=1}^{n} x_j$ and $s = \sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n$.

Notice that

$$s_{n+1} - s_n = (x_1 + x_2 + \cdots + x_{n+1}) - (x_1 + x_2 + \cdots + x_n) = x_{n+1}$$

Therefore we have

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}(s_{n+1} - s_n) = \lim_{n\to\infty} s_{n+1} - \lim_{n\to\infty} s_n = s - s = 0$$

It follows that $\lim_{n\to\infty} x_n = 0$ (x_n small for large n). $\square$

Notice that it is vital that $s \in \mathbb{R}$ (a finite sum). If we allowed series to have sum ∞ (or $-\infty$ either) then we could not make sense of $s - s$.

The point of the next example is that the theorem tells us a property that every convergent series has, but it does not work in reverse. If $\lim_{n\to\infty} x_n = 0$ it may still be the case that the series $\sum_{n=1}^{\infty} x_n$ fails to be summable.

Example 1.3.5. The series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is *not convergent*,

(This is a famous example and has a special name — it is called the *harmonic series*. [It occurred to me to wonder why? The answer is related to music somehow, according to Wikipedia.])

Proof. We can show by induction that the partial sums

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

satisfy

$$s_{2^n} \geq 1 + \frac{n}{2}$$

For $n = 1$ we have $s_2 = 1 + \frac{1}{2}$. (So the inequality holds for $n = 1$.)

Assuming as an inductive hypothesis that $s_{2^n} \geq 1 + \frac{n}{2}$ we have

$$\begin{aligned}
s^{2^{n+1}} &= \sum_{j=1}^{2^{n+1}} \frac{1}{j} = \sum_{j=1}^{2^n} \frac{1}{j} + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j} \\
&= s_{2^n} + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j} \\
&> 1 + \frac{n}{2} + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{2^{n+1}} \\
&= 1 + \frac{n}{2} + (2^{n+1} - 2^n)\frac{1}{2^{n+1}} = 1 + \frac{n}{2} + \frac{1}{2} = 1 + \frac{n+1}{2}
\end{aligned}$$

It is clear from this that the sequence of partial sums $s_1, s_2, s_3, \ldots$ is not bounded above and the limit $\lim_{n\to\infty} s_n$ cannot be any finite s. (One way to argue this is that if there was a finite limit s then for all large enough n we would have $|s_n - s| < 1$, so $s-1 < s_n < s+1$. But if we choose m large enough we would have $s+1 < (m+1)/2$ and

$$s + 1 < \frac{m+1}{2} < s_{2^m}$$

This shows for sure that the limit of the partial sums is not finite.) □

Proposition 1.3.6. *(Link to monotone sequences)*
If $\sum_{n=1}^{\infty} x_n$ is a series of nonnegative terms then its sequence of partial sums is monotone increasing.

Proof. As usual we will write $s_n = \sum_{j=1}^{n} x_j = x_1 + x_2 + \cdots + x_n$ for the n^{th} partial sum.
We can then observe that $s_{n+1} - s_n = x_{n+1} \geq 0$ and so $s_n \leq s_{n+1}$ always.
This says that $(s_n)_{n=1}^{\infty}$ is monotone increasing. □

Theorem 1.3.7. *If $\sum_{n=1}^{\infty} x_n$ is a series of nonnegative terms then it is convergent if and only if its sequence of partial sums is bounded above.*

Proof. Since this is an if and only if statement there are two things to show.
$\Rightarrow$: Assume first that $\sum_{n=1}^{\infty} x_n$ is convergent. Write $s_n = \sum_{j=1}^{n} x_j$ and $s = \lim_{n\to\infty} s_n$. Taking $\varepsilon = 1$ in the definition of limit, there is N so that

$$n > N \Rightarrow |s_n - s| < 1 \Rightarrow s_n < s + 1$$

For $n \leq N$ we have $s_n \leq s_{N+1} < s + 1$ also holds and so $s_n \leq s + 1$ for all n. (Recall that we have a series of nonnegative terms.) That says that the number $s+1$ is an upper bound for the partial sums. So the partial sums are bounded above.
$\Leftarrow$: Suppose now $\sum_{n=1}^{\infty} x_n$ is a series (of nonnegative terms) with partial sums bounded above. By Proposition 1.3.6 the sequence of partial sums is monotone increasing and so we know it has a limit (by Theorem 1.2.3, since it is a bounded monotone sequence). □

Theorem 1.3.8 (Comparison test for series of positive terms)**.** *Suppose $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are two series with nonnegative terms ($x_n \geq 0$ and $y_n \geq 0$) and suppose $x_n \leq y_n$ for each n. Then*

(a) if the larger series $\sum_{n=1}^{\infty} y_n$ converges then so does the smaller series $\sum_{n=1}^{\infty} x_n$;

(b) if the smaller series $\sum_{n=1}^{\infty} x_n$ does not converge then neither does the larger one $\sum_{n=1}^{\infty} y_n$.

Proof. Notice that the partial sums of the smaller series are smaller than those of the larger: $\sum_{j=1}^{n} x_j \leq \sum_{j=1}^{n} y_j$ for each n. Using Theorem 1.3.7, if $\sum_{n=1}^{\infty} y_n$ converges then there is $u \in \mathbb{R}$ so that $\sum_{j=1}^{n} y_j \leq u$ for all n. But then $\sum_{j=1}^{n} x_j \leq u$ also and so $\sum_{n=1}^{\infty} x_n$ converges (using Theorem 1.3.7). That shows (a). And (b) follows from (a). □