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Theorem 1.1.10. *If $f\colon I \to \mathbb{R}$ is a function defined on an interval $I \subseteq \mathbb{R}$ and if $a \in I$, then f is continuous at a if and only if*

> *for all sequences $(x_n)_{n=1}^\infty$ of terms $x_n \in I$ satisfying $\lim_{n\to\infty} x_n = a$ we have $\lim_{n\to\infty} f(x_n) = f(a)$*

Proof. We will not give the proof of this. It is rather similar to the proof of the previous theorem. We don't need to rule out $x_n = a$ in this case. □

By the way, the result would hold with an interval I replaced by any subset $S \subseteq \mathbb{R}$ but maybe you have not seen continuity defined for domains that are not intervals.

There is a version of Theorem 1.1.9 to express other kinds of limits, such as $\lim_{x\to\infty} f(x) = \ell$ in terms of limits of sequences, but for that we would need to define what it means to have $\lim_{n\to\infty} x_n = \infty$ (and $\lim_{n\to\infty} x_n = -\infty$ as well). These are not too hard to define but we will leave out the formal definitions.

1.2 Monotone sequences

Definition 1.2.1. A sequence $(x_n)_{n=1}^\infty$ (in $\mathbb{R}$) is called *monotone increasing* if $x_n \leq x_{n+1}$ for each n.

So that means $x_1 \leq x_2 \leq x_3 \leq \cdots$ and it implies that $x_n \leq x_m$ if $n < m$.

A sequence $(x_n)_{n=1}^\infty$ is called *monotone decreasing* if $x_n \geq x_{n+1}$ for each n.

A sequence is called *monotone* is it is either monotone increasing *or* monotone decreasing.

A sequence $(x_n)_{n=1}^\infty$ is called *strictly monotone decreasing* if $x_n > x_{n+1}$ for each n.

A sequence $(x_n)_{n=1}^\infty$ is called *strictly monotone increasing* if $x_n < x_{n+1}$ for each n.

A sequence is called *strictly monotone* is it is either strictly monotone increasing *or* strictly monotone decreasing.

Examples 1.2.2. (i) If $x_n = 1/n$, then $(x_n)_{n=1}^\infty = \left(\frac{1}{n}\right)_{n=1}^\infty$ is a monotone decreasing sequence, in fact a strictly monotone decreasing one.

The limit 0 of this sequence coincides with the greatest lower bound for the set of values, $\{x_n : n \in \mathbb{N}\} = \{1, 1/2, 1/3, \ldots\}$.

(ii) On the other hand $x_n = 1 - \frac{1}{n^2}$ is a (strictly) monotone increasing sequence which has $\lim_{n\to\infty} x_n = 1 =$ the least upper bound of $\{x_n : n \in \mathbb{N}\} = \{0, 3/4, 8/9, \ldots\}$.

(iii) If $(x_n)_{n=1}^\infty$ is any monotone increasing sequence, then $(-x_n)_{n=1}^\infty$ is monotone decreasing.

Using this one can prove things about monotone decreasing sequences once one has proved the similar fact about increasing sequences.

(iv) The sequence $((-1)^n)_{n=1}^\infty$ is not monotone.

Theorem 1.2.3 (a variation on the least upper bound principle)**.** *If $(x_n)_{n=1}^\infty$ is any monotone increasing sequence with an upper bound in $\mathbb{R}$, then $\lim_{n\to\infty} x_n$ exists in $\mathbb{R}$.*

Proof. The idea is to show that the limit must be the least upper bound of the set $\{x_n : n \in \mathbb{N}\}$.

Recall that an upper bound for the set is a number $u \in \mathbb{R}$ so that $x_n \leq u$ for each n. A least upper bound U is what it says — an upper bound so that no smaller number is an upper bound.

We won't give the proof very carefully but the idea is that $\lim_{n\to\infty} x_n = U =$ the least upper bound (which must exist because of the least upper bound principle). Given $\varepsilon > 0$, $U - \varepsilon$ is too small to be an upper bound and so there is N with $x_N > U - \varepsilon$. For $n > N$ we have $x_n \geq x_N > U - \varepsilon$ but also $x_n \leq U$, which implies $|x_n - U| < \varepsilon$. □

Corollary 1.2.4. *If $(x_n)_{n=1}^\infty$ is any monotone decreasing sequence with a lower bound in $\mathbb{R}$, then $\lim_{n\to\infty} x_n$ exists in $\mathbb{R}$.*

Remark 1.2.5. Monotone sequences are then somewhat simpler that general sequences, at least in terms of whether they have limits or not. If they are bounded they do have limits.

If a monotone increasing sequence $(x_n)_{n=1}^\infty$ is not bounded (above — it is bounded below by x_1) then in fact $\lim_{n\to\infty} x_n = \infty$. We skipped the formal definition of infinite limits but maybe you can understand it without the formalities. This tells us how to explain the long run behaviour of the sequence fairly well. We don't consider sequences with infinite limits as being as well-behaved as those with ordinary finite limits.

The term 'convergent sequence' is usually reserved for those with finite limits.

1.3 Series

A *series* (of real numbers) is in fact just a sequence $(x_n)_{n=1}^\infty$ but we use the word 'series' when we are thinking of adding up the infinite list of numbers, to try and make sense of

$$x_1 + x_2 + x_3 + \cdots$$

Such an infinite sum does not make sense without a definition of what it might mean. There is no theoretical difficulty with any finite sum (though there could be a practical difficulty if the number of terms is really huge).

Notation 1.3.1. *We write $\sum_{n=1}^\infty x_n$ to indicate a sequence $(x_n)_{n=1}^\infty$ considered as a series.*

The partial sums *of a series $\sum_{n=1}^\infty x_n$ are the finite sums*

$$s_n = x_1 + x_2 + \cdots + x_n = \sum_{j=1}^n x_j$$

Definition 1.3.2. A series $\sum_{n=1}^\infty x_n$ is called *convergent* if the sequence of its partial sums has a finite limit, that is if $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{j=1}^n x_j$ exists in $\mathbb{R}$.