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About MA1132 — Advanced calculus

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Refer to `http://www.maths.tcd.ie/~richardt/MA1132` for more information.

Prerequisites: MA1123 (Analysis on the real line I), MA1111 (Linear Algebra I)

2 lectures and 1 tutorial per week

Assessment: Regular aassignments and tutorial work. In class exams Friday February 24 and (to be confirmed) Monday April 2nd.

The entire grade will be based on continuous assessment.

For those who do not complete the module satisfactorily, there will be a supplemental examination.

Outline

Sequences (definition and basic results on convergence). Series (definition of the sum, series of positive terms, absolute convergence, tests for convergence). Power series and (use of) Taylor's theorem.

Differentiation of curves, tangent lines in 2 or 3 dimensions. Graphical represenation of functions of 2 or 3 variables. Partial derivatives, gradients, directional derivatives, tangent planes to graphs and level surfaces. Linear approximation for functions of 2 or 3 variables, chain rule.

Linear and exact differential equations.

Double and triple integrals, computation via iterated integrals (Fubini theorem). Double integrals in polar coordinates.

Integrals of vector fields (along curves). Greens theorem.

1 Limits of sequences, sums of series

1.1 Sequences

Normally we think of a sequence (infinite sequence in our case) as an infinite list $x_1, x_2, x_3, \ldots$. We will be dealing first with sequences of (real) numbers but it does make sense to consider sequences of other things later (vectors, matrices, functions for example).

We can define a sequence as a function, though that is not the usual way to think of them.

Definition 1.1.1. By a *sequence* (in $\mathbb{R}$) we mean a function $x\colon \mathbb{N} \to \mathbb{R}$.

I use $\mathbb{N} = \{1, 2, 3, \ldots\}$ for the positive integers (or natural numbers — some people include 0 as a natural number but I don't). Since a function $x\colon \mathbb{N} \to \mathbb{R}$ is a rule that associates a value $x(n)$ to each n in the domain $\mathbb{N}$, it follows that we can think of a sequence as a list of numbers $x(1), x(2), x(3), \ldots$. It is usual though to write sequences with subscript notation — so x_n instead of $x(n)$. We typically describe a sequence by the notation

$$(x_n)_{n=1}^\infty$$

rather than the more elongated $x_1, x_2, x_3, \ldots$.

We refer to x_n as the n^{th} *term* of the sequence.

It will sometimes be convenient to write a sequence as a list like $t_0, t_1, t_2, \ldots$ starting at 0, abbreviated $(t_n)_{n=0}^\infty$. That could be a function with domain $\{0\} \cup \mathbb{N}$ or we could say it was the sequence $(x_n)_{n=1}^\infty$ with n^{th} term $x_n = t_{n-1}$.

Examples 1.1.2. Typically sequences are described by a rule giving the terms, such as

$$\begin{aligned} x_n &= (-1)^n \\ y_n &= \sin n \\ u_n &= \begin{cases} \dfrac{3n^2+2n-5}{n^2+n-1} & \text{if } n \text{ is odd} \\ \dfrac{3n^2+7n-2}{n^2-n-1} & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

So the sequence $(x_n)_{n=1}^\infty$ is $-1, 1, -1, 1, \ldots$ (alternating minus ones and ones). We could describe that sequence without giving any name x_n for the terms as the sequence $((-1)^n)_{n=1}^\infty$ and the second example as $(\sin n)_{n=1}^\infty$.

Our main concern will be limits of sequences (when they have limits). Informally $\lim_{n\to\infty} x_n = \ell$ means that x_n will certainly be close to ℓ for all large n. More precisely we define it be saying that once we decide how close we mean when we say 'close to ℓ' there must be a suitable interpretation of largeness for n to make the previous assertion true.

Definition 1.1.3. If $(x_n)_{n=1}^\infty$ is a sequence (of real numbers) and $\ell \in \mathbb{R}$ then we say that the sequence *converges to* ℓ (and write $\lim_{n\to\infty} x_n = \ell$) if the following ε-N criterion is satisfied:

given any $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$n \in \mathbb{N}, n > N \Rightarrow |x_n - \ell| < \varepsilon.$$

Remark 1.1.4. Notice that the N will (almost always) depend on ε and one could write N_ε for N to emphasise that. So the ε-N criterion would then read

given any $\varepsilon > 0$ there is some $N_\varepsilon \in \mathbb{N}$ such that

$$n \in \mathbb{N}, n > N_\varepsilon \Rightarrow |x_n - \ell| < \varepsilon.$$

Thinking back over the various kinds of limits you have seen for limits of functions, such

$$\lim_{x\to a} f(x) = \ell, \qquad \lim_{x\to\infty} f(x) = \ell, \qquad \lim_{x\to a} f(x) = \infty, \qquad \lim_{x\to\infty} f(x) = \infty,$$

the definition for the limit of a sequence is most similar to the definition for $\lim_{x\to\infty} f(x) = \ell$. In that case the setting is that $f(x)$ must at least make sense on some interval (x_0, ∞) and the criterion is

given any $\varepsilon > 0$ there is some $R \in \mathbb{N}$ such that

$$x \in \mathbb{R}, x > R \Rightarrow |f(x) - \ell| < \varepsilon.$$

So the difference with a sequence is that the domain we consider is only the natural numbers, not a half line.

Examples 1.1.5. (i) (Limits of constant sequences) If $x_n = k \in \mathbb{R}$ for all n, then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} k = k$$

(ii)

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

These need to be checked directly from the definition. (That is not so hard — try it for yourself.)

We can work out a fair number of basic limits using these two examples plus the following theorem.

Theorem 1.1.6. *Suppose $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are two convergent sequences with $\lim_{n\to\infty} x_n = \ell_1$ and $\lim_{n\to\infty} y_n = \ell_2$.*

(i) (Multiples) For $k \in \mathbb{R}$ a fixed number,

$$\lim_{n\to\infty} k x_n = k\ell_1$$

(ii) (Sums)

$$\lim_{n\to\infty} x_n + y_n = \ell_1 + \ell_2$$

(iii) (Products)

$$\lim_{n\to\infty} x_n y_n = \ell_1 \ell_2$$

(iv) (Quotients) Provided (y_n is never 0 and) $\ell_2 \neq 0$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\ell_1}{\ell_2}$$

This fact about limits of sums, products and quotients is proved in a way that mimics the proofs of the same results for limits of functions. We omit the proofs.

One thing to notice is that the limit of a sequence depends only on the terms for large n. A formal way to express that is that if you change just a finite number of terms of a sequence then you don't change the limit. Also if the sequence was not convergent, changing a finite number of terms will still give a sequence with no limit.

In the result about lmits of quotients, the hypothesis that $\ell_2 \neq 0$ is somehow more important than the one about $y_n \neq 0$ for all n. The reason is that if $\ell_2 \neq 0$ then it must be that $y_n \neq 0$ for all n large enough (large enough that $|y_n - \ell_2|$ is smaller than $|\ell_2|$). So x_n/y_n will make sense for all large n if $\ell_2 \neq 0$. Of course then it is a problem that the sequnce of fractions $(x_n/y_n)_{n=1}^\infty$ might not make real sense if y_n is ever 0. But we could make some arbitrary convention about what to replace x_n/y_n by when $y_n = 0$ and then the result would hold as long as $\ell_2 \neq 0$.

Examples 1.1.7. (i) An example of a sequence with no limit is the alternating sequence $((-1)^n)_{n=1}^\infty$.

Proof. This needs a proof (based on the definition 1.1.3), which I leave as an exercise for you. □

(ii) If x_n is given by a rational function where the degree of the numerator is no larger than the degree of the denominator, we can find $\lim_{n\to\infty} x_n$ by dividing above and below by the highest power of n in the denominator and using the theorem 1.1.6 and the examples 1.1.5 repeatedly.

For instance

$$\begin{aligned}
\lim_{n\to\infty} \frac{4n^3 - n^2 + 11n + 102}{7n^3 + n - 3} &= \lim_{n\to\infty} \frac{4 - \frac{1}{n} + 11\frac{1}{n^2} + 102\frac{1}{n^3}}{7 + \frac{1}{n^2} - 3\frac{1}{n^3}} \\
&= \frac{\lim_{n\to\infty} 4 - \frac{1}{n} + 11\frac{1}{n^2} + 102\frac{1}{n^3}}{\lim_{n\to\infty} 7 + \frac{1}{n^2} - 3\frac{1}{n^3}}
\end{aligned}$$

using Theorem 1.1.6 (iv) — which will be justified once we can show that the limits in the numerator and denominator do make sense. Looking first at the numerator, we know (by 1.1.5) $\lim_{n\to\infty} 4 = 4$ and $\lim_{n\to\infty} 1/n = 0$, hence $\lim_{n\to\infty} -11/n = 0$ (by 1.1.6 (i) and so $\lim_{n\to\infty}(4 - 11/n) = 4$ (by 1.1.6 (ii)). Next $\lim_{n\to\infty} 1/n^2 = \lim_{n\to\infty}(1/n)(1/n) = 0$ (by 1.1.6 (iii)) and by the same result on limits of products $\lim_{n\to\infty} 1/n^3 = \lim_{n\to\infty}(1/n^2)(1/n) = 0$. Using 1.1.6 (i) and 1.1.6 (ii) again, we deduce

$$\lim_{n\to\infty}\left(4 - 1\frac{1}{n} + 11\frac{1}{n^2}\right) = \lim_{n\to\infty}\left(4 - 1\frac{1}{n}\right) + \lim_{n\to\infty} 11\frac{1}{n^2} = 4 + 11(0) = 4$$

and finally we have the limit of the numerator

$$\lim_{n\to\infty}\left(4 - 1\frac{1}{n} + 11\frac{1}{n^2} + 120\frac{1}{n^3}\right) = \lim_{n\to\infty}\left(4 - 1\frac{1}{n} + 11\frac{1}{n^2}\right) + \lim_{n\to\infty} 120\frac{1}{n^3} = 4 + 120(0) = 4$$

Similarly, but compressing the various steps, we have the limit of the denominator

$$\lim_{n\to\infty}\left(7 + \frac{1}{n^2} - 3\frac{1}{n^3}\right) = 7 + 0 - 3(0) = 7$$

and the limit of the original sequence is then justfied as a limit of a quotient

$$\lim_{n\to\infty}\frac{4n^3 - n^2 + 11n + 102}{7n^3 + n - 3} = \frac{\lim_{n\to\infty} 4 - \frac{1}{n} + 11\frac{1}{n^2} + 102\frac{1}{n^3}}{\lim_{n\to\infty} 7 + \frac{1}{n^2} - 3\frac{1}{n^3}} = \frac{4}{7}$$

We can describe limits of functions $\lim_{x\to a} f(x) = \ell$ using sequences, as follows. Recall that for $\lim_{x\to a} f(x) = \ell$ to make sense we need $f(x)$ to be defined near $x = a$ but not at $x = a$ itself. The domain of $f(x)$ must include a 'punctured open interval' about a, which means a set $(a_0, a_1) \setminus \{a\} = (a_0, a) \cup (a, a_1)$ with $a_0 < a < a_1$.

Theorem 1.1.8. *Let $f\colon S \to \mathbb{R}$ be a function defined on a subset $S \subseteq \mathbb{R}$ that contains a punctured open interval around a point $a \in \mathbb{R}$. Suppose $\ell \in \mathbb{R}$.*

Then $\lim_{x\to a} f(x) = \ell$ holds if and only if the following sequence criterion is valid

> *for all sequences $(x_n)_{n=1}^\infty$ of terms $x_n \in S \setminus \{a\}$ satisfying $\lim_{n\to\infty} x_n = a$ we have $\lim_{n\to\infty} f(x_n) = \ell$*

Proof. There are two separate things to prove because this is an 'if and only if' theorem. First we prove that if $\lim_{x\to a} f(x) = \ell$ then the sequence criterion holds. Next we prove (in a separate argument) that if the sequence criterion holds then $\lim_{x\to a} f(x) = \ell$.

$\Rightarrow$: Suppose now $\lim_{x\to a} f(x) = \ell$. Consider an arbitrary sequence $(x_n)_{n=1}^\infty$ of terms $x_n \in S \setminus \{a\}$ satisfying $\lim_{n\to\infty} x_n = a$.

[First we will explain what we are going to do somewhat imprecisely. We know x_n is close to a (but not equal to a) for n big enough and if x is close enough to a (yet diferent from a) then $f(x)$ will be close to ℓ. So if n is big enough, x_n will be close to a and $f(x_n)$ close to ℓ. That is $\lim_{n\to\infty} f(x_n) = \ell$. What we need though is to make these terms 'close' and 'large' more precise.]

According to Definition 1.1.3 we should start with $\varepsilon > 0$ fixed and show that we can find N so that

$$n > N \Rightarrow |f(x_n) - \ell| < \varepsilon$$

So we now suppose that $\varepsilon > 0$ is given.

According to the ε-δ definition of what $\lim_{x \to a} f(x) = \ell$ means, we know we can find $\delta > 0$ so that

$$x \in \mathbb{R}, 0 < |x - a| < \delta \Rightarrow |f(x) - \ell| < \varepsilon$$

Applying the definition 1.1.3 of what $\lim_{n\to\infty} x_n = a$ means (applied with $\delta > 0$ as the 'epsilon') there is $N = N_\delta$ so that

$$n > N \Rightarrow |x_n - a| < \delta.$$

Since we also know $x_n \neq a$ for each n we see that

$$n > N \Rightarrow 0 < |x_n - a| < \delta \Rightarrow |f(x_n) - \ell| < \varepsilon$$

Since we can find such N no matter which $\varepsilon > 0$ we begin with, we have shown $\lim_{n\to\infty} f(x_n) = \ell$.

$\Leftarrow$: Suppose now that the sequence criterion holds.

To make the notation of the proof easier, we need to work with a symmetric punctured open interval about a that is contained in S, the domain of f. Since S contains $(a_0, a_1) \cup (a, a_1)$ for some $a_0 < a < a_1$, if we take $\alpha = \min(a - a_0, a_1 - a)$ then $\alpha > 0$ and S contains $(a - \alpha, a + \alpha) \setminus \{a\} = \{x \in \mathbb{R} : 0 < |x - a| < \alpha\}$.

To prove $\lim_{x\to a} f(x) = \ell$, we show that it cannot fail to be true. If it fails to be the case that the limit is ℓ, it means that there is some $\varepsilon > 0$ where no $\delta > 0$ has the property

$$x \in \mathbb{R}, 0 < |x - a| < \delta \Rightarrow |f(x) - \ell| < \varepsilon$$

Since then $\delta = \alpha/n$ does not work (and all x with $0 < |x - a| < \alpha/n$ are in the domain of f), there must be x_n with

$$0 < |x_n - a| < \frac{\alpha}{n} \text{ and } |f(x_n) - \ell| \geq \varepsilon$$

If we pick such x_n for each $n = 1, 2, 3, \ldots$, we end up with a sequence $(x_n)_{n=1}^{\infty}$ so that $x_n \in S \setminus \{a\}$ for all n, $\lim_{n\to\infty} x_n = a$ but $(f(x_n))_{n=1}^{\infty}$ does not converge to ℓ. That contadicts our assumption and so we must be able to find $\delta > 0$ to suit each $\varepsilon > 0$. We have shown that $\lim_{x\to a} f(x) = \ell$ is true. □

Example 1.1.9. $\lim_{n\to\infty} 1/\sqrt{n} = 0$ because $\lim_{x\to 0} \sqrt{|x|} = 0$ and $\lim_{n\to\infty} 1/n = 0$. Similarly $\lim_{n\to\infty} \sin(1/n) = 0$.

MA1132 (R. Timoney) January 17, 2012[revised to change ℓ to a is a few places in the statement of the last theorem and its proof.]