MA1311 (Advanced Calculus) Tutorial sheet 9

[December 2 – 3, 2010]

Name: Solutions

1. Consider the function $f(x, y, z) = x^3y + y^2z$ and suppose there is a parametric curve $\mathbf{p}(t) = (x(t), y(t), z(t))$ in $\mathbb{R}^3$ satisfying $\mathbf{p}(2) = (2, 4, -1)$ and $\mathbf{p}'(2) = (1, 0, -1)$. Find

$$\frac{d}{dt}\,|_{t=2}\, f(\mathbf{p}(t)).$$

[This involves the chain rule in 3 variables.] Also give the linear approximation for $f(x, y, z)$ centered on $(2, 4, -1)$.

Solution: From the chain rule

$$\frac{d}{dt} f(x(t), y(t), z(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

we see we need the partial derivatives of f. They are

$$\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2y \\ \frac{\partial f}{\partial y} &= x^3 + 2yz \\ \frac{\partial f}{\partial z} &= y^2 \end{aligned}$$

In fact we need their values at $(x, y, z) = \mathbf{p}(2) = (2, 4, -1)$ to get the derivative we want at $t = 2$.

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) |_{(2,4,-1)} = (48, 8 - 8, 16) = (48, 0, 16)$$

So

$$\frac{d}{dt} f(x(t), y(t), z(t)) = (48, 0, 16) \cdot (1, 0, -1) = 48 + 0 - 16 = 32.$$

The linear approximation formula is

$$f(x, y, z) \cong f(2, 4, -1) + f_x(2, 4, -1)(x - 2) + f_y(2, 4, -1)(y - 4) + f_z(2, 4, -1)(z + 1)$$

for (x, y, z) near $(2, 4, -1)$. We have already worked out the values of the partial derivatives and we just need $f(2, 4, -1) = 32 - 16 = 16$. So the linear approximation formula is

$$f(x, y, z) \cong 16 + 48(x - 2) + 16(z + 1).$$

2. Find

$$\frac{d}{dx}\int_{x^3}^{2} \cos t^2\,dt$$

Solution:

$$\begin{aligned}
\frac{d}{dx}\int_{x^3}^{2} \cos t^2\,dt &= \frac{d}{dx}\left(-\int_{2}^{x^3} \cos t^2\,dt\right)\\
&= -\cos(x^3)^2(3x^2) = -3x^2\cos x^6
\end{aligned}$$

where we have used the chain rule and the fundamental theorem, which tells us that

$$\frac{d}{du}\int_{2}^{u} \cos t^2\,dt = \cos u^2.$$

3. (a) Find $\displaystyle\int x\sec^2(3x^2)\,dx$.

Solution: This can be done by substituting $u = 3x^2$, so that $du = 6x\,dx$ or $dx = du/(6x)$. So

$$\begin{aligned}
\int x\sec^2(3x^2)\,dx &= \int x\sec^2 u\,\frac{du}{6x}\\
&= \int \frac{1}{6}\sec^2 u\,du\\
&\quad \text{Note: all } x \text{ and } dx \text{ gone now and only } u \text{ and } du \text{ remain}\\
&= \frac{1}{6}\tan u + C\\
&= \frac{1}{6}\tan(3x^2) + C
\end{aligned}$$

(b) Find $\displaystyle\int_{-1/2}^{0} \sin(\pi x)e^{\cos(\pi x)}\,dx$.

Solution: Here we substitute $u = \cos(\pi x)$ so that $du = -\pi\sin(\pi x)\,dx$. For $x = -1/2$ we get $u = \cos(-\pi/2) = 0$ and for $x = 0$ we get $u = \cos 0 = 1$. So we have

$$\begin{aligned}
\int_{-1/2}^{0} \sin(\pi x)e^{\cos(\pi x)}\,dx &= \int_{u=0}^{u=1} \sin(\pi x)e^u\,\frac{du}{-\pi\sin(\pi x)}\\
&= \int_{u=0}^{u=1} -\frac{1}{\pi}e^u\,du\\
&= \left[-\frac{1}{\pi}e^u\right]_{u=0}^{u=1}\\
&= -\frac{1}{\pi}e^1 - \left(-\frac{1}{\pi}\right) = \frac{1-e}{\pi}
\end{aligned}$$

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