MA1311 (Advanced Calculus) Tutorial sheet 8
[November 25 – 26, 2010]

Name: Solutions

For this sheet consider the function

$$f(x,y,z) = \frac{x^2e^xy + xe^yz}{x+y+z}$$

1. Find ∇f (evaluated at an unspecified point (x,y,z)).

   Solution:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{(2xe^xy + x^2e^xy + e^yz)(x+y+z) - (x^2e^xy + xe^yz)}{(x+y+z)^2} \\
\frac{\partial f}{\partial y} &= \frac{(x^2e^x + xe^yz)(x+y+z) - (x^2e^xy + xe^yz)}{(x+y+z)^2} \\
\frac{\partial f}{\partial z} &= \frac{xe^y(x+y+z) - (x^2e^xy + xe^yz)}{(x+y+z)^2}
\end{aligned}$$

$$\begin{aligned}
\nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
&= \left(\frac{(2xe^xy + x^2e^xy + e^yz)(x+y+z) - (x^2e^xy + xe^yz)}{(x+y+z)^2},\right. \\
&\qquad \frac{(x^2e^x + xe^yz)(x+y+z) - (x^2e^xy + xe^yz)}{(x+y+z)^2}, \\
&\qquad \left.\frac{xe^y(x+y+z) - (x^2e^xy + xe^yz)}{(x+y+z)^2}\right)
\end{aligned}$$

2. Find the equation of the tangent plane to the surface

$$f(x,y,z) = \frac{2e}{3}$$

   at the point $(1,1,1)$.

   Solution: We are getting the tangent plane to a level surface $f(x,y,z) = 2e/3$. So the normal vector to the tangent plane is the gradient of f evaluated at the point $(1,1,1)$.

$$\nabla f \mid_{(1,1,1)} = \left(\frac{(4e)(3) - 2e}{3^2}, \frac{(2e)(3) - 2e}{3^2}, \frac{3e - 2e}{3^2}\right) = \left(\frac{10e}{9}, \frac{4e}{9}, \frac{e}{9}\right).$$

   For the tangent plane we get

$$\frac{10e}{9}(x-1) + \frac{4e}{9}(y-1) + \frac{e}{9}(z-1) = 0$$

   and we could simplify this by dividing by $e/9$ to get the equivalent equation

$$10(x-1) + 4(y-1) + (z-1) = 0.$$

3. Find the direction $\mathbf{u} = (u_1, u_2, u_3)$ for which the directional derivative $D_{\mathbf{u}}f(1, 1, -1)$ is as small as possible. Also, what is that smallest possible value of $D_{\mathbf{u}}f(1, 1, -1)$?

   Solution: What we need is $\mathbf{u}$ to be the unit vector in the direction opposite to $\nabla f \mid_{(1,1,-1)}$ and the smallest possible value is $-\|\nabla f \mid_{(1,1,-1)} \|$.

$$\nabla f \mid_{(1,1,-1)} = \left(\frac{(2e + e - e)(1) - 0}{1^2}, 0 - 0, \frac{e(1) - 0}{1^2} \right) = (2e, 0, e)$$

   So

$$\mathbf{u} = -\frac{1}{\|\nabla f \mid_{(1,1,-1)} \|} \nabla f \mid_{(1,1,-1)} = \frac{-1}{e\sqrt{5}}(2e, 0, e) = \left(-\frac{2}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}} \right)$$

   and the corresponing smallest value of the directional derivative is

$$D_{\mathbf{u}}f(1, 1, -1) = -\|\nabla f \mid_{(1,1,-1)} \| = -\sqrt{5}e$$

Richard M. Timoney