MA1311 (Advanced Calculus) Tutorial sheet 7
[November 17 – 18, 2010]

Name: Solutions.

Let $(x_0, y_0) = (1, -1)$ and

$$f(x,y) = \frac{e^{xy}}{x^4 + y^4}$$

for these problems.

1. Find the directional derivative $D_{\mathbf{u}} f(x_0, y_0)$ for the function $f(x,y)$, the point (x_0, y_0), and $\mathbf{u} = (u_1, u_2)$ any unit vector.

   Solution: We could work out the directional derivative from the definition $D_{\mathbf{u}} f(x_0, y_0) = \frac{d}{dt} |_{t=0} f((x_0, y_0) + t\mathbf{u})$ but it is simpler to rely on the formula

   $$D_{\mathbf{u}} f = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2$$

   or the version evaluated at $(x_0, y_0) = (1, -1)$

   $$D_{\mathbf{u}} f(x_0, y_0) = D_{\mathbf{u}} f(1, -1) = f_x(1, -1) u_1 + f_y(1, -1) u_2$$

   We need the partial derivatives

   $$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{y e^{xy}(x^4 + y^4) - 4x^3 e^{xy}}{(x^4 + y^4)^2} \\ \frac{\partial f}{\partial y} &= \frac{x e^{xy}(x^4 + y^4) - 4y^3 e^{xy}}{(x^4 + y^4)^2} \\ \frac{\partial f}{\partial x} |_{(1,-1)} &= \frac{-2e^{-1} - 4e^{-1}}{4} \\ &= \frac{-3}{2e} \\ \frac{\partial f}{\partial x} |_{(1,-1)} &= \frac{2e^{-1} + 4e^{-1}}{4} \\ &= \frac{3}{2e} \end{aligned}$$

   Thus

   $$D_{\mathbf{u}} f(1, -1) = -\frac{3}{2e} u_1 + \frac{3}{2e} u_2$$

2. For the same $f(x,y)$, the same (x_0, y_0) and the corresponding point (x_0, y_0, z_0) on the graph $z = f(x,y)$, find parametric equations for the line which is tangent to the graph at (x_0, y_0, z_0) and perpendicular to the x-axis.

Solution: The line will be tangent to the curve where the graph $z = f(x, y)$ intersects the plane $x = x_0$, and so tangent to the curve

$$\begin{aligned} x &= x_0 \\ y &= y_0 + t \\ z &= f(x_0, y_0 + t) \end{aligned}$$

at $t = 0$. The tangent vector to that curve is

$$\left(0, 1, \frac{\partial f}{\partial y} |_{(x_0,y_0)}\right) = (0, 1, \frac{3}{2e}).$$

So the line is

$$\begin{aligned} x &= x_0 \\ y &= y_0 + t \\ z &= z_0 + t(\frac{3}{2e}) \end{aligned}$$

and $z_0 = f(x_0, y_0) = \frac{e^{-1}}{1+1} = 1/(2e)$. Thus the answer is

$$\begin{aligned} x &= 1 \\ y &= -1 + t \\ z &= \frac{1}{2e} + \frac{3t}{2e}. \end{aligned}$$

3. For the same $f(x, y)$, the same (x_0, y_0) and the corresponding point (x_0, y_0, z_0) on the graph $z = f(x, y)$, find the normal vector to the tangent plane to the graph $z = f(x, y)$ at (x_0, y_0, z_0). And then find the equation of that plane.

   Solution: The normal vector to the tangent plane is

$$\left(\frac{\partial f}{\partial x} |_{(x_0,y)}, \frac{\partial f}{\partial x} |_{(x_0,y)}, -1\right) = (f_x(x_0, y_0), f_y(x_0, y_0), -1)$$

   and we have those values already. So the normal vector is

$$\left(\frac{-3}{2e}, \frac{3}{2e}, -1\right) = -\frac{3}{2e}\mathbf{i} + \frac{3}{2e}\mathbf{j} - \mathbf{k}$$

   The equation of the tangent plane must then have the form

$$\alpha x + \beta y + \gamma z = c$$

   where $(\alpha, \beta, \gamma) = \left(\frac{-3}{2e}, \frac{3}{2e}, -1\right)$. So the equation has the form

$$-\frac{3}{2e}x + \frac{3}{2e}y - z = c$$

Also $(x_0, y_0, z_0) = (1, -1, 1/(2e))$ must satisfy the equation and so we have

$$-\frac{3}{2e} - \frac{3}{2e} - \frac{1}{2e} = c$$

or $-7/(2e) = c$. That means the equation is

$$-\frac{3}{2e}x + \frac{3}{2e}y - z = -\frac{7}{2e}$$

or we might prefer

$$3x - 3y + 2ez = 7$$

4. Find the linear approximation formula for $f(x, y)$ centered at $(x, y) = (x_0, y_0)$.

   Solution: The formula we want is

$$f(x, y) \cong f(x_0, y_0) + \frac{\partial f}{\partial x} \mid_{(x_0,y_0)} (x - x_0) + \frac{\partial f}{\partial y} \mid_{(x_0,y_0)} (y - y_0).$$

   We need $f(x_0, y_0) = f(1, -1) = 1/(2e)$ and the values of the partials evaluated at $(1, -1)$ (which we have already).

   So we end up with

$$f(x, y) \cong \frac{1}{2e} - \frac{3}{2e}(x - 1) + \frac{3}{2e}(y + 1)$$

   (which we expect to be a good approximation for (x, y) close to $(1, -1)$).

5. Find the gradient vector $\nabla f \mid_{(x_0,y_0)}$ (evaluated at (x_0, y_0)).

   Solution: We know $\nabla f = (\partial f/\partial x, \partial f/\partial y)$ and we already had the values of the partials evaluated at $(x_0, y_0) = (1, -1)$. We get

$$\nabla f \mid_{(x_0,y_0)} = \nabla f \mid_{(1,-1)} = \left(-\frac{3}{2e}, \frac{3}{2e}\right) = -\frac{3}{2e}\mathbf{i} + \frac{3}{2e}\mathbf{j}$$

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