MA1311 (Advanced Calculus) Tutorial/Exercise sheet 6
[November 4 – 5, 2010]

Name: Solutions

1. Find the partial derivatives with respect to x and y evaluated at $(x_0, y_0) = (2, -2)$ for

$$f(x,y) = \frac{x\cos(\pi y)}{x^2+y^2}$$

Solution: We could do this

$$\begin{aligned}
f(x, y_0) &= f(x, -2) \\
&= \frac{x\cos(-2\pi)}{x^2+(-2)^2} \\
&= \frac{x}{x^2+4} \\
\frac{\partial f}{\partial x}\,|_{(x,-2)} &= \frac{d}{dx}f(x,y_0) = \frac{d}{dx}\frac{x}{x^2+4} \\
&= \frac{(x^2+4)-x(2x)}{(x^2+4)^2} \\
&= \frac{4-x^2}{(x^2+4)^2} \\
\frac{\partial f}{\partial x}\,|_{(2,-2)} &= 0
\end{aligned}$$

and this

$$\begin{aligned}
f(x_0, y) &= f(2, y) \\
&= \frac{2\cos(\pi y)}{4+y^2} \\
\frac{\partial f}{\partial y}\,|_{(2,y)} &= \frac{d}{dy}f(x_0,y) = \frac{d}{dy}\frac{2\cos(\pi y)}{4+y^2} \\
&= \frac{-2\pi\sin(\pi y)(4+y^2) - 2\cos(\pi y)(2y)}{(x^2+4)^2} \\
\frac{\partial f}{\partial x}\,|_{(2,-2)} &= \frac{0+8}{8^2} = \frac{1}{8}
\end{aligned}$$

But this is how we would usually do it:

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\cos(\pi y)(x^2+y^2) - x\cos(\pi y)(2x)}{(x^2+y^2)^2} \\
&= \frac{(x^2+y^2-2x^2)\cos(\pi y)}{(x^2+y^2)^2} \\
&= \frac{(y^2-x^2)\cos(\pi y)}{(x^2+y^2)^2} \\
\frac{\partial f}{\partial y} &= \frac{-\pi x\sin(\pi y)(x^2+y^2) - x\cos(\pi y)(2y)}{(x^2+y^2)^2} \\
&= \frac{-\pi x(x^2+y^2)\sin(\pi y) - 2xy\cos(\pi y)}{(x^2+y^2)^2} \\
\frac{\partial f}{\partial x}\,|_{(2,-2)} &= 0 \\
\frac{\partial f}{\partial y}\,|_{(2,-2)} &= \frac{0+8}{8^2} = \frac{1}{8}
\end{aligned}
$$

2. Find the partials $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ evaluated at (x_0, y_0) (which means $\frac{\partial f}{\partial x}\,|_{(x_0,y_0)}= f_x(x_0,y_0)$ and $\frac{\partial f}{\partial y}\,|_{(x_0,y_0)}= f_y(x_0,y_0)$) for the following cases.

   (a) $f(x,y) = e^{(x+2y-3)}$, $(x_0,y_0) = (-1,1)$
   Solution:

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= e^{(x+2y-3)}\frac{\partial}{\partial x}(x+2y-3) \\
&= e^{(x+2y-3)} \\
\frac{\partial f}{\partial y} &= e^{(x+2y-3)}\frac{\partial}{\partial y}(x+2y-3) \\
&= 2e^{(x+2y-3)} \\
\frac{\partial f}{\partial x}\,|_{(-1,1)} &= e^{-2} \\
\frac{\partial f}{\partial y}\,|_{(-1,1)} &= 2e^{-2}
\end{aligned}
$$

(b) $f(x,y) = \sqrt{x^2+y^2}$, $(x_0, y_0) = (-3, 4)$
Solution:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{1}{2\sqrt{x^2+y^2}}(2x) \\
&= \frac{x}{\sqrt{x^2+y^2}} \\
\frac{\partial f}{\partial y} &= \frac{1}{2\sqrt{x^2+y^2}}(2y) \\
&= \frac{y}{\sqrt{x^2+y^2}} \\
\frac{\partial f}{\partial x}\,|_{(-3,4)} &= \frac{-3}{5} \\
\frac{\partial f}{\partial y}\,|_{(-3,4)} &= \frac{4}{5}
\end{aligned}$$

(c) $f(x,y) = \cos^2(3\pi x^2 - \pi y^4)$, $(x_0, y_0) = (1, 0)$
Solution:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2\cos(3\pi x^2 - \pi y^4)(-\sin(3\pi x^2 - \pi y^4))(6\pi x) \\
&= -12\pi x\cos(3\pi x^2 - \pi y^4)\sin(3\pi x^2 - \pi y^4) \\
\frac{\partial f}{\partial y} &= 2\cos(3\pi x^2 - \pi y^4)(-\sin(3\pi x^2 - \pi y^4))(-4\pi y^3) \\
&= 8\pi y^3\cos(3\pi x^2 - \pi y^4)\sin(3\pi x^2 - \pi y^4) \\
\frac{\partial f}{\partial x}\,|_{(1,0)} &= -12\pi\cos(3\pi)\sin(3\pi) = 0 \\
\frac{\partial f}{\partial y}\,|_{(1,0)} &= 0
\end{aligned}$$

3. Find the partials $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ (as functions of (x, y)) for the following cases

(a) $f(x,y) = 3x^2y - 8x^3y^2 + 5x^4y - 17$
Solution:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 6xy - 24x^2y^2 + 20x^3y \\
\frac{\partial f}{\partial y} &= 3x^2 - 16x^3y + 5x^4
\end{aligned}$$

(b) $f(x,y) = \dfrac{x-y}{x+xy+y-1}$

Solution:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{1(x+xy+y-1)-(x-y)(1+y)}{(x+xy+y-1)^2}\\
&= \frac{y^2+2y-1}{(x+xy+y-1)^2}\\
\frac{\partial f}{\partial y} &= \frac{(-1)(x+xy+y-1)-(x-y)(x+1)}{(x+xy+y-1)^2}\\
&= \frac{-x^2-2x+1}{(x+xy+y-1)^2}
\end{aligned}$$

(c) $f(x,y) = \dfrac{xe^{xy}-x^2\sin(x+y)}{x+e^{(x+y)}}$

Solution:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{(e^{xy}+xye^{xy}-2x\sin(x+y)-x^2\cos(x+y))(x+e^{(x+y)})}{(x+e^{(x+y)})^2}\\
&\quad -\frac{(xe^{xy}-x^2\sin(x+y))(1+e^{(x+y)})}{(x+e^{(x+y)})^2}\\
\frac{\partial f}{\partial y} &= \frac{(x^2e^{xy}-x^2\cos(x+y))(x+e^{(x+y)})-(xe^{xy}-x^2\sin(x+y))e^{(x+y)}}{(x+e^{(x+y)})^2}
\end{aligned}$$

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