**MA1311 (Advanced Calculus) Tutorial sheet 5**
[October 28 – 29, 2010]

**Name:** Solutions

1. Find $\frac{d}{dx}\sin^{-1}x+\cos^{-1}x$. Can you explain your answer?

   *Solution:*

   $$\frac{d}{dx}\sin^{-1}x+\cos^{-1}x=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}=0 \quad (\text{for } -1<x<1).$$

   This means that $\sin^{-1}x+\cos^{-1}x=c=$ constant. Taking $x=0$ gives $\sin^{-1}0+\cos^{-1}0=0+\pi/2=\pi/2=c$. Thus $\sin^{-1}x+\cos^{-1}x=\pi/2$ for $-1<x<1$.

   Indeed, $\sin(\pi/2-\theta)=\cos(\theta)$ always and if $0\le\theta\le\pi$ with $\cos\theta=x$, then we have $\theta=\cos^{-1}x$. But also $-\pi/2\le\pi/2-\theta\le\pi/2$ and $\sin(\pi/2-\theta)=x$, from which we find $\pi/2-\theta=\sin^{-1}x$. So $\sin^{-1}x+\cos^{-1}x=(\pi/2-\theta)+\theta=\pi/2$ (for $-1\le x\le 1$).

2. Find parametric equations for the tangent line to the parametric curve

   $$\mathbf{x}(t)=(t\cosh t, t\sinh t, t)$$

   at the point where $t=0$.

   *Solution:* We need $\mathbf{x}'(0)$ (which is a vector parallel to the tangent line) and $\mathbf{x}(0)=(0,0,0)$ (which is a point on the line).

   $$\mathbf{x}'(t)=\frac{d\mathbf{x}}{dt}=(\cosh t+t\sinh t, \sinh t+t\cosh t, 1)$$

   and so

   $$\mathbf{x}'(0)=(1+0,0+0,1)=(1,0,1).$$

   The line has parametric equations

   $$\begin{cases} x &= 0+1t\\ y &= 0+0t\\ z &= 0+1t\end{cases}$$

   or

   $$\begin{cases} x &= t\\ y &= 0\\ z &= t\end{cases}$$

3. Find

- $\dfrac{d}{dx}\sin^{-1}(2x^3)$,
  *Solution:*
  $$\frac{d}{dx}\sin^{-1}(2x^3) = \frac{1}{\sqrt{1-(2x^3)^2}}(6x^2) = \frac{6x^2}{\sqrt{1-4x^6}}$$
- $\dfrac{d}{dx}\tan^{-1}(8x)$,
  *Solution:*
  $$\frac{d}{dx}\tan^{-1}(8x) = \frac{1}{1+(8x)^2}(8) = \frac{8}{1+64x^2}$$
- $\dfrac{d}{dx}x^{(5^x)}$,
  *Solution:*
  $$x^{(5^x)} = x^{\left(\left(e^{\ln 5}\right)^x\right)} = \left(e^{\ln x}\right)^{\left(e^{x\ln 5}\right)} = e^{(\ln x)e^{x\ln 5}}$$
  Hence
  $$\frac{d}{dx}x^{(5^x)} = e^{(\ln x)e^{x\ln 5}}\frac{d}{dx}((\ln x)e^{x\ln 5}) = x^{(5^x)}\left(\frac{1}{x}e^{x\ln 5} + (\ln x)e^{x\ln 5}(\ln 5)\right)$$
  or
  $$\frac{d}{dx}x^{(5^x)} = x^{(5^x)}\left(\frac{5^x}{x} + (\ln 5)(\ln x)5^x\right)$$
- $\dfrac{d}{dx}\sinh^{-1}(x/7)$
  *Solution:*
  $$\begin{aligned}\frac{d}{dx}\sinh^{-1}(x/7) &= \frac{1}{\sqrt{1+(x/7)^2}}\frac{1}{7}\\ &= \frac{1}{7\sqrt{1+\frac{x^2}{49}}}\\ &= \frac{1}{\sqrt{49\left(1+\frac{x^2}{49}\right)}}\\ &= \frac{1}{\sqrt{49+x^2}}\end{aligned}$$
- and $f(x)$ if $f'(x) = \dfrac{1}{1-2x^2}$ (for $-1/\sqrt{2} < x < 1/\sqrt{2}$) and $f(0) = 2$.

  *Solution:* Looking at the result of the derivative of $\tan^{-1}(5x)$ above and recalling $(d/dx)\tanh^{-1}x = 1/(1-x^2)$, we can see that
  $$\frac{d}{dx}\tanh^{-1}(\sqrt{2}x) = \frac{1}{1-(\sqrt{2}x)^2}\sqrt{2} = \frac{\sqrt{2}}{1-2x^2}$$

and so

$$f'(x) = \frac{d}{dx}\left(\frac{1}{\sqrt{2}}\tanh^{-1}(\sqrt{2}x)\right)$$

or

$$\frac{d}{dx}\left(f(x) - \frac{1}{\sqrt{2}}\tanh^{-1}(\sqrt{2}x)\right) = 0.$$

Hence

$$f(x) - \frac{1}{\sqrt{2}}\tanh^{-1}(\sqrt{2}x) = c = \text{ constant}$$

Taking $x = 0$ we get

$$2 - \frac{1}{\sqrt{2}}\tanh^{-1} 0 = c$$

so that $2 = c$. We end up with

$$f(x) = \frac{1}{\sqrt{2}}\tanh^{-1}(\sqrt{2}x) + 2$$

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