MA1311 (Advanced Calculus) Tutorial sheet 4

[October 21 – 22, 2010]

Name: Solutions

1. Show that

$$\frac{e^{x^2}}{e^x}$$

is strictly monotone increasing for $x > 1/2$.

Solution: It will be enough to show the derivative is strictly positive for $x > 1/2$.

$$\frac{d}{dx}\frac{e^{x^2}}{e^x} = \frac{d}{dx}e^{x^2-x} = (2x-1)e^{x^2-x} = 2(x-1/2)e^{x^2-x}.$$

Since $e^{x^2-x} > 0$ always, this derivative has the same sign as $x - 1/2$. So

$$\frac{d}{dx}\frac{e^{x^2}}{e^x} > 0 \qquad \left(x > \frac{1}{2}\right)$$

and it follows that $\frac{e^{x^2}}{e^x}$ is strictly monotone increasing for $x > 1/2$.

2. Find all the solutions to the differential equation

$$\frac{dy}{dx} - 2y = \frac{e^{2x}}{x} \qquad (x > 0).$$

Solution: We can use an integrating factor e^{-2x} to get

$$e^{-2x}\frac{dy}{dx} - 2e^{-2x}y = \frac{e^{-2x}e^{2x}}{x} = \frac{e^0}{x} = \frac{1}{x}.$$

So we have

$$\frac{d}{dx}(e^{-2x}y) = \frac{1}{x}.$$

We know

$$\frac{1}{x} = \frac{d}{dx}\ln x \qquad (\text{for } x > 0)$$

and so we have

$$\frac{d}{dx}(e^{-2x}y) = \frac{d}{dx}\ln x$$

or

$$\frac{d}{dx}(e^{-2x}y - \ln x) = 0.$$

From that we have $e^{-2x}y - \ln x = c =$ constant. So $e^{-2x}y = c + \ln x$ or

$$y = (c + \ln x)e^{2x}.$$

So every solution must be of this form (for some $c \in \mathbb{R}$), and we can check that every y of this form solves the differential equation. The fact that they all solve follows because we can reverse all the steps we used to find y, or we can just check it:

$$\frac{dy}{dx} - 2y = \frac{1}{x}e^{2x} + 2(c + \ln x)e^{2x} - 2(c + \ln x)e^{2x} = \frac{e^{2x}}{x}$$

3. Find the inverse function for $y = f(x) = 4x^3 - 1$.

   Solution: We should solve $f(x) = y$ for x (in terms of y). We have

   $$4x^3 - 1 = y \qquad \text{or} \qquad 4x^3 = y + 1 \qquad \text{or} \qquad x^3 = \frac{y+1}{4}.$$

   So the solution is

   $$x = \left(\frac{y+1}{4}\right)^{1/3} = f^{-1}(y)$$

   We would usually use x as the independent variable and say that

   $$f^{-1}(x) = \left(\frac{x+1}{4}\right)^{1/3}$$

   is the inverse function.

4. Show that (for $n \in \mathbb{N}$)

   $$\frac{x}{(\ln x)^n}$$

   is strictly monotone increasing for $x > e^n$.

   Solution: We should look at the derivative

   $$\frac{d}{dx}\frac{x}{(\ln x)^n} = \frac{(\ln x)^n - x\left(n(\ln x)^{n-1}\frac{1}{x}\right)}{(\ln x)^{2n}} = \frac{(\ln x)^n - n(\ln x)^{n-1}}{(\ln x)^{2n}} = \frac{\ln x - n}{(\ln x)^{n+1}}.$$

   For $x > e^n$ we have $\ln x > \ln e^n = n$ (because $\ln x$ is strictly monotone increasing on $(0, \infty)$). So certainly $\ln x > 0$ but also $\ln x - n > 0$. So

   $$\frac{d}{dx}\frac{x}{(\ln x)^n} > 0 \qquad \text{for } x > e^n.$$

   Thus $x/(\ln x)^n$ is strictly monotone increasing for $x > e^n$.

   [Remark: This is actually rather closely related to the fact that e^y/y^n is strictly monotone increasing for $y > n$ (which we had in the notes). Take $y = \ln x$ so that $e^y = x$ and then

   $$\frac{x}{(\ln x)^n} = \frac{e^y}{y^n}.$$

   Since $y = \ln x$ increases with x, and e^y/y^n increases with y for $y > n$, we could conclude the same thing by a different method. For $y > n$ we need $x > e^n$.]

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