MA1311 (Advanced Calculus) Tutorial sheet 3
[October 14 – 15, 2010]

Name: Solutions

1. Find $\dfrac{dy}{dx}$ for $y = \dfrac{(x+1)\tan(2x)}{(x^4+1)^{1/4}}$

   Solution:

$$\begin{aligned}
\frac{dy}{dx} & \\
&= \frac{(\tan(2x)+(x+1)\sec^2(2x)(2))(x^4+1)^{1/4}-(x+1)\tan(2x)\left(\frac{1}{4}(x^4+1)^{-3/4}(4x^3)\right)}{(x^4+1)^{1/2}}\\
&= \frac{(\tan(2x)+2(x+1)\sec^2(2x))(x^4+1)^{1/4}-x^3(x+1)\tan(2x)(x^4+1)^{-3/4}}{(x^4+1)^{1/2}}\\
&= \frac{(\tan(2x)+2(x+1)\sec^2(2x))(x^4+1)-x^3(x+1)\tan(2x)}{(x^4+1)^{5/4}}\\
&= \frac{(1-x^3)\tan(2x)+2(x+1)(x^4+1)\sec^2(2x)}{(x^4+1)^{5/4}}
\end{aligned}$$

2. Find the derivative dy/dx if $y = f(x)$ is a function that satisfies the equation

$$x^2\cos^2 y + y^2\sin^2 x = \frac{\pi^2}{16}$$

   Using the general result you get, and assuming $f(\pi/4) = \pi/4$, find $f'(\pi/4)$.

   Solution: Differentiating both sides of the equation with respect to x, treating y as a function of x and using the product rule and chain rule multiple times, we get

$$2x\cos^2 y + x^2\left(2\cos y(-\sin y)\frac{dy}{dx}\right) + 2y\frac{dy}{dx}\sin^2 x + y^2(2\sin x\cos x) = 0$$

$$\begin{aligned}
(-2x^2\cos y\sin y + 2y\sin^2 x)\frac{dy}{dx} &= -2x\cos^2 y - 2y^2\sin x\cos x\\
\frac{dy}{dx} &= -\frac{2x\cos^2 y + 2y^2\sin x\cos x}{-2x^2\cos y\sin y + 2y\sin^2 x}\\
&= \frac{x\cos^2 y + y^2\sin x\cos x}{x^2\cos y\sin y - y\sin^2 x}
\end{aligned}$$

Evaluating that at $(x, y) = (\pi/4, \pi/4)$, gives

$$\begin{aligned} f'(\pi/4) &= \frac{dy}{dx}\,|_{x=\pi/4} \\ &= \frac{(\pi/4)\cos^2(\pi/4) + (\pi/4)^2\sin(\pi/4)\cos(\pi/4)}{(\pi/4)^2\cos(\pi/4)\sin(\pi/4) - (\pi/4)\sin^2(\pi/4)} \\ &= \frac{(\pi/4)(1/2) + (\pi^2/16)(1/2)}{(\pi^2/16)(1/2) - (\pi/4)(1/2)} \\ &= \frac{4\pi + \pi^2}{\pi^2 - 4\pi} \\ &= \frac{4+\pi}{\pi - 4} \end{aligned}$$

3. Show that $f(x) = x\cos x - \sin x$ is strictly monotone decreasing for $0 < x < \pi/2$.

   Solution: This will follow if $f'(x) < 0$ for $0 < x < \pi/2$. We compute

   $$f'(x) = \cos x + x(-\sin x) - \cos x = -x\sin x.$$

   For $0 < x < \pi/2$ we have $x > 0$, $\sin x > 0$ and so $f'(x) < 0$.

   It follows that $f(x)$ is strictly monotone decreasing for $0 < x < \pi/2$.

4. Find the critical points for $f(x) = x - \sin x$.

   Solution: That means the solutions of $f'(x) = 0$. We have $f'(x) = 1 - \cos x$ and so we need the solutions to $\cos x = 1$. These are $x = 0$ and values of x different from that by multiples of 2π. So $x = 2n\pi$ with $n \in \mathbb{Z}$.

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