MA1311 (Advanced Calculus) Tutorial sheet 2
[October 7 – 8, 2010]

Name: Solutions

1. For the (rational) function $f(x) = \dfrac{3x^2 + 15x - 42}{2x^2 - 9x + 10}$, find

$$\lim_{x \to 2} f(x) \qquad \text{and} \qquad \lim_{x \to 0} f\left(\frac{1}{x}\right)$$

Solution: For $\lim_{x\to 2} f(x)$ we see that the denominator becomes 0 at $x = 2$. So does the numerator. Thus there must be a common factor $x - 2$ which can be cancelled.

Here we do the factoring by long division of polynomials (although you could surely do it other ways).

$$\begin{array}{r|rrrrr} & 3x & + & 21 & & \\ \hline x-2 & 3x^2 & + & 15x & - & 42 \\ & 3x^2 & - & 6x & & \\ \hline & & & 21x & - & 42 \\ & & & 21x & - & 42 \\ \hline & & & & & 0 \end{array}$$

$$\begin{array}{r|rrrrr} & 2x & - & 5 & & \\ \hline x-2 & 2x^2 & - & 9x & + & 10 \\ & 2x^2 & - & 4x & & \\ \hline & & & -5x & + & 10 \\ & & & -5x & + & 10 \\ \hline & & & & & 0 \end{array}$$

So

$$f(x) = \frac{3x^2 + 15x - 42}{2x^2 - 9x + 10} = \frac{(x-2)(3x+21)}{(x-2)(2x-5)} = \frac{3x+21}{2x-5}$$

(for $x \neq 2$), and

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{3x+21}{2x-5} = \frac{27}{-1} = -27$$

For $\lim_{x\to 0} f\left(\frac{1}{x}\right)$ we can rewrite

$$\begin{aligned}
f\left(\frac{1}{x}\right) &= \frac{3\left(\frac{1}{x}\right)^2 + 15\frac{1}{x} - 42}{2\left(\frac{1}{x}\right)^2 - 9\frac{1}{x} + 10} \\
&= \frac{\left(3\left(\frac{1}{x}\right)^2 + 15\frac{1}{x} - 42\right)x^2}{\left(2\left(\frac{1}{x}\right)^2 - 9\frac{1}{x} + 10\right)x^2} \\
&= \frac{3 + 15x - 42x^2}{2 - 9x + 10x^2} \\
&= \frac{3}{2}
\end{aligned}$$

2. Find $\dfrac{dy}{dx}$ for $y = \dfrac{x^3 - 5tx^2 + cx + 3}{(x-t)^2}$ (where t and c are constants).

   Solution: Using the quotient rule first

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\left(\frac{d}{dx}(x^3 - 5tx^2 + cx + 3)\right)(x-t)^2 - (x^3 - 5tx^2 + cx + 3)\left(\frac{d}{dx}(x-t)^2\right)}{((x-t)^2)^2} \\
&= \frac{(3x^2 - 10tx + c)(x-t)^2 - (x^3 - 5tx^2 + cx + 3)(2(x-t)(1))}{(x-t)^4} \\
&\qquad \text{(using the chain rule for } \tfrac{d}{dx}(x-t)^2\text{)} \\
&= \frac{(3x^2 - 10tx + c)(x-t)^2 - 2(x-t)(x^3 - 5tx^2 + cx + 3)}{(x-t)^4} \\
&= \frac{(3x^2 - 10tx + c)(x-t) - 2(x^3 - 5tx^2 + cx + 3)}{(x-t)^3} \\
&= \frac{x^3 - 3tx^2 + 10t^2x - cx - ct - 6}{(x-t)^3}
\end{aligned}$$

   (Probably multiplying this out doesn't make it any much more tidy.)

3. Find the linear approximation to the graph $y = 4x^3 - 8x^2 + 5x - 21$ centered at the point $x = 1$.

   Solution: The formula we want is $f(x) \cong f(1) + f'(1)(x-1)$ and we want this for $f(x) = 4x^3 - 8x^2 + 5x - 21$.

   We need $f(1) = 4 - 8 + 5 - 21 = -20$, $f'(x) = 12x^2 - 16x + 5$ and $f'(1) = 1$. So we get

$$4x^3 - 8x^2 + 5x - 21 \cong -20 + 1(x-1) = -20 + (x-1) \qquad \text{(for } x \text{ near } 1\text{).}$$

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