MA1311 (Advanced Calculus) Tutorial sheet 12
[Due Monday January 17th, 2011]

Name: Solutions

1. Express the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

as an iterated integral (*i.e.* set up the triple integral in terms of single integrals, but do not evaluate it).

Solution: Denoting the ellipsoid by D, its volume is given by

$$\iiint_D 1\,dx\,dy\,dz.$$

We integrate first with respect to z, keeping (x, y) fixed. The limits for z arise from expressing the equation for the surface of the ellipsoid in the form

$$z = \pm c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

Having integrated with respect to z, we must then take the double integral of the result over the vertical projection (or "shadow") of the ellipsoid onto the xy-plane. This projection is, in this case, the section of the ellipsoid through $z = 0$, or the inside of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In this way we arrive at the answer

$$\int_{x=-a}^{x=a} \left(\int_{y=-b\sqrt{1-x^2/a^2}}^{y=b\sqrt{1-x^2/a^2}} \left(\int_{z=-c\sqrt{1-x^2/a^2-y^2/b^2}}^{z=c\sqrt{1-x^2/a^2-y^2/b^2}} 1\,dz \right) dy \right) dx.$$

We could equally do the integrals in another order. If we did the indegral dx first, then dy and finally dz we would get

$$\int_{z=-c}^{z=c} \left(\int_{y=-b\sqrt{1-z^2/c^2}}^{y=b\sqrt{1-z^2/c^2}} \left(\int_{x=-a\sqrt{1-y^2/b^2-z^2/c^2}}^{x=a\sqrt{1-y^2/b^2-z^2/c^2}} 1\,dx \right) dy \right) dz.$$

All the orders give integrals that are painful to evaluate.

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2. Evaluate

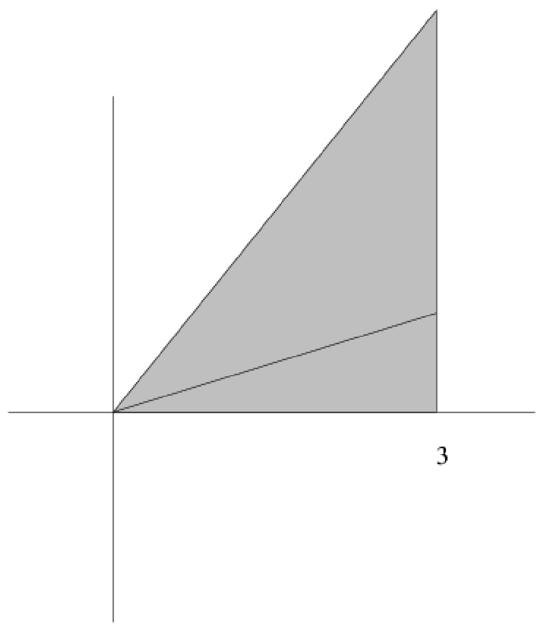
$$\int_0^3 \int_{y=0}^{y=\sqrt{3}x} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx$$

by making a change of variables to polar coordinates. [Hint: Sketch the region first. Then do the dr integral first, before $d\theta$. $\int sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$]

Solution: This iterated integral is the same (by Fubini's theorem) as the double integral

$$\iint_R \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy$$

where R is the triangle in the plane bounded by the x-axis, the line $x = 3$ and the line $y = \sqrt{3}x$.



The limit $x = 3$ is $r\cos\theta = 3$ or $r = 3/\cos\theta$ and the angle is $\pi/3$ (because $\tan(\pi/3) = \sqrt{3}$).

Changing this integral to polar coordinates, and remembering that $dx\, dy = r\, dr\, d\theta$, we get

$$\begin{aligned}
\int_{\theta=0}^{\theta=\pi/3} \left(\int_{r=0}^{r=3/\cos\theta} \frac{1}{r} r\, dr \right) d\theta &= \int_{\theta=0}^{\theta=\pi/3} [r]_{r=0}^{r=3/\cos\theta}\, d\theta \\
&= \int_{\theta=0}^{\theta=\pi/3} \frac{3}{\cos\theta}\, d\theta \\
&= \int_{\theta=0}^{\theta=\pi/3} 3\sec\theta\, d\theta \\
&= [3\ln|\sec\theta + \tan\theta|]_0^{\pi/3} \\
&= 3\ln(2+\sqrt{3})
\end{aligned}$$

3. Find the mass of a solid object occupying the region in space bounded by the coordinate planes and the plane $x + y + z = 2$ if its density function is $\delta(x, y, z) = x^2$. [Hint: The mass is the triple integral of the density over the region. Calculations are easier if you leave the dx integral to last.]

   Solution: We know that the answer is

$$\text{Mass} = \iiint dm = \iiint \delta(x, y, z)\, dx\, dy\, dz$$

   with the triple integral extending over the object.

   That gives

$$\begin{aligned}
&\int_{x=0}^{2} \left(\int_{y=0}^{2-x} \left(\int_{z=0}^{z=2-x-y} x^2\, dz \right) dy \right) dx \\
&= \int_{x=0}^{2} \left(\int_{y=0}^{2-z} \left[x^2 z \right]_{z=0}^{z=2-x-y} dx \right) dy \\
&= \int_{x=0}^{2} \left(\int_{y=0}^{2-x} x^2(2 - x - y)\, dy \right) dx \\
&= \int_{x=0}^{2} \left[x^2((2 - x)y - y^2/2) \right]_{y=0}^{2-x} dx \\
&= \int_{x=0}^{2} x^2((2 - x)^2 - (2 - x)^2/2) - 0\, dx \\
&= \int_{x=0}^{2} x^2(2 - x)^2/2\, dx = \frac{1}{2} \int_{x=0}^{2} x^2(4 - 4x + x^2)\, dx \\
&= \frac{1}{2} \int_{x=0}^{2} 4x^2 - 4x^3 + x^4\, dx \\
&= \frac{1}{2} \left[(4/3)x^3 - x^4 + x^5/5 \right]_{x=0}^{2} \\
&= \frac{1}{2} \left(\frac{32}{3} - 16 + \frac{32}{5} - 0 \right) = \frac{8}{15}
\end{aligned}$$

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