MA1311 (Advanced Calculus) Tutorial sheet 11

[December 16 – 17, 2010]

Name: Solutions

1. For $R = [2,3] \times [-1,1]$ (the rectangle $\{(x,y) \in \mathbb{R}^2 : 2 \le x \le 3, -1 \le y \le 1\}$) and $f(x,y) = \cos\left(\frac{\pi}{2}x\right) + \sin\left(\frac{\pi}{3}y\right)$, find $\iint_R f(x,y)\,dx\,dy$

   Solution:

$$
\begin{aligned}
\iint_R f(x,y)\,dx\,dy &= \int_{y=-1}^{1}\left(\int_{x=2}^{3}\cos\left(\frac{\pi}{2}x\right) + \sin\left(\frac{\pi}{3}y\right)\,dx\right)dy \\
&= \int_{y=-1}^{1}\left[\frac{2}{\pi}\sin\left(\frac{\pi}{2}x\right) + x\sin\left(\frac{\pi}{3}y\right)\right]_{x=2}^{3} dy \\
&= \int_{y=-1}^{1}\frac{2}{\pi}\sin\frac{3\pi}{2} + 3\sin\left(\frac{\pi}{3}y\right) - \left(\frac{2}{\pi}\sin\pi + 2\sin\left(\frac{\pi}{3}y\right)\right)dy \\
&= \int_{y=-1}^{1} -\frac{2}{\pi} + \sin\left(\frac{\pi}{3}y\right)dy \\
&= \left[-\frac{2}{\pi}y - \frac{3}{\pi}\cos\left(\frac{\pi}{3}y\right)\right]_{y=-1}^{1} \\
&= -\frac{2}{\pi} - \frac{3}{\pi}\cos\frac{\pi}{3} - \left(\frac{2}{\pi} - \frac{3}{\pi}\cos\left(-\frac{\pi}{3}\right)\right) \\
&= -\frac{4}{\pi}
\end{aligned}
$$

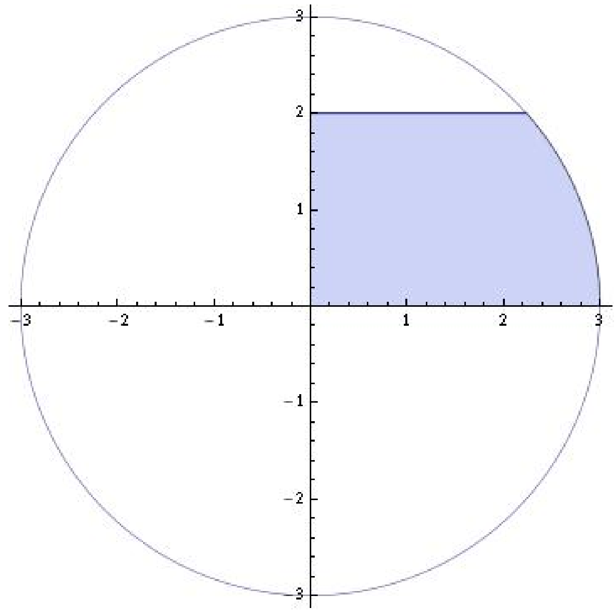
2. Find and graph the region R in $\mathbb{R}^2$ so that the interated integral

$$
\int_{y=0}^{2}\int_{x=0}^{x=\sqrt{9-y^2}} f(x,y)\,dx\,dy = \iint_R f(x,y)\,dx\,dy
$$

   Solution:

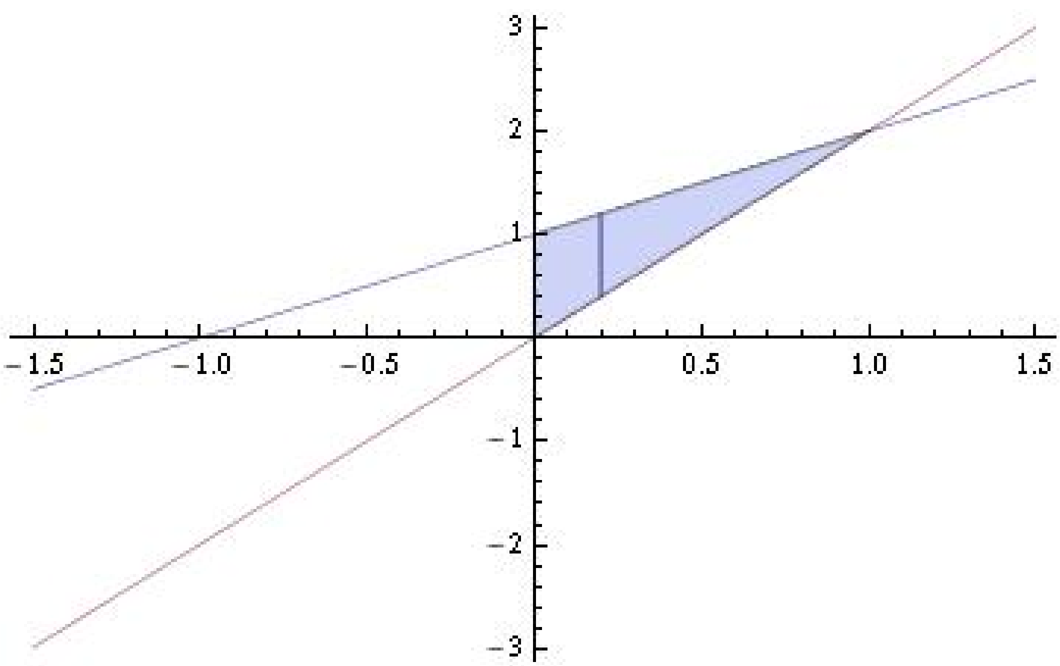
$$
\begin{aligned}
R &= \{(x,y) \in \mathbb{R}^2 : 0 \le x \le \sqrt{9-y^2}, 0 \le y \le 2\} \\
&= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 9, x \ge 0, 0 \le y \le 2\}
\end{aligned}
$$

   so that R is part of the first quadrant inside the disk of radius 3 around the origin, below the line $y = 2$.



3. Find $\iint_R x^3 + y^2\, dx\, dy$ when R is the region in the plane bounded by the y-axis and the lines $y = 2x$, $y = x + 1$.

   Solution: A picture of R is good to have



   The lines cross where $2x = x + 1$, or $x = 1$.

In this case it is easier to integrate with respect to y first (keeping x fixed).

$$\begin{aligned}
\iint_R x^3 + y^2\,dx\,dy &= \int_{x=0}^{x=1} \left(\int_{y=2x}^{y=x+1} x^3 + y^2\,dy \right) dx \\
&= \int_{x=0}^{x=1} \left[x^3 y + \frac{y^3}{3} \right]_{y=2x}^{y=x+1} dx \\
&= \int_{x=0}^{x=1} x^3(x+1-2x) + \frac{1}{3}(x+1)^3 - \frac{8}{3}x^3\,dx \\
&= \int_0^1 x^3 - x^4 + \frac{1}{3}(x+1)^3 - \frac{8}{3}x^3\,dx \\
&= \left[\frac{x^4}{4} - \frac{x^5}{5} + \frac{1}{12}(x+1)^4 - \frac{2}{3}x^4 \right]_0^1 \\
&= \frac{1}{4} - \frac{1}{5} + \frac{16}{12} - \frac{2}{3} - \frac{1}{12} \\
&= \frac{19}{30}
\end{aligned}$$

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