MA1311 (Advanced Calculus) Tutorial/Exercise sheet 10

[due 17.50am, Tuesday, December 14, 2010]

Brief solutions

1. Find $\int (x^2 - 3x)e^{4x}\,dx$

   Solution: Integration by parts (twice) does this.

$$\int (x^2-3x)e^{4x}\,dx \qquad \text{Let } \begin{array}{rcl} u &=& x^2-3x \\ du &=& (2x-3)\,dx \end{array} \qquad \begin{array}{rcl} dv &=& e^{4x}\,dx \\ v &=& \frac{1}{4}e^{4x} \end{array}$$

$$\begin{aligned}
\int (x^2-3x)e^{4x}\,dx &= \int u\,dv \\
&= uv - \int v\,du \\
&= (x^2-3x)\left(\frac{1}{4}e^{4x}\right) - \int \frac{1}{4}e^{4x}(2x-3)\,dx \\
&\quad \text{Let } \begin{array}{rcl} U &=& 2x-3 \\ dU &=& 2\,dx \end{array} \quad \begin{array}{rcl} dV &=& \frac{1}{4}e^{4x}\,dx \\ V &=& \frac{1}{16}e^{4x} \end{array} \\
&= \frac{x^2-3x}{4}e^{4x} - \int U\,dV \\
&= \frac{x^2-3x}{4}e^{4x} - \left(UV - \int V\,dU\right) \\
&= \frac{x^2-3x}{4}e^{4x} - UV + \int V\,dU \\
&= \frac{x^2-3x}{4}e^{4x} - \frac{2x-3}{16}e^{4x} + \int \frac{1}{16}e^{4x}2\,dx \\
&= \frac{x^2-3x}{4}e^{4x} - \frac{2x-3}{16}e^{4x} + \frac{1}{32}e^{4x} + C \\
&= \frac{8x^2-28x+7}{32}e^{4x} + C
\end{aligned}$$

2. Find $\int \frac{x^2-3x}{2x^3-9x^2+21}\,dx$

   Solution: Substitution of $u = 2x^3 - 9x^2 + 21$, $du = (6x^2 - 18x)\,dx = 6(x^2 - 3x)\,dx$

works here

$$\begin{aligned}
\int \frac{x^2-3x}{2x^3-9x^2+21}\,dx &= \int \frac{x^2-3x}{u}\,\frac{du}{6(x^2-3x)} \\
&= \int \frac{1}{6}\frac{1}{u}\,du \\
&= \frac{1}{6}\ln|u|+C \\
&= \frac{1}{6}\ln|2x^3-9x^2+21|+C
\end{aligned}$$

3. Find $\int \cos^4(3x)\,dx$

   Solution: Method for even powers of $\cos\theta$ is to use $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$\begin{aligned}
\int \cos^4(3x)\,dx &= \int (\cos^2(3x))^2\,dx \\
&= \int \left(\frac{1}{2}(1+\cos(6x)\right)^2\,dx \\
&= \int \frac{1}{4}(1+2\cos(6x)+\cos^2 6x)\,dx \\
&= \int \frac{1}{4}\left(1+2\cos(6x)+\frac{1}{2}\cos(12x)\right)\,dx \\
&= \int \left(\frac{3}{8}+\frac{1}{2}\cos(6x)+\frac{1}{8}\cos(12x)\right)\,dx \\
&= \frac{3}{8}x+\frac{1}{12}\sin(6x)+\frac{1}{96}\sin(12x)+C
\end{aligned}$$

4. Find $\int_0^{\pi/4} \sec^2 x\,dx$

   Solution:

$$\int_0^{\pi/4} \sec^2 x\,dx = [\tan x]_0^{\pi/4} = \tan(\pi/4) = \tan 0 = 1-0 = 1$$

5. Find $\int \tan^2 x\,dx$

   Solution:

$$\int \tan^2 x\,dx = \int \sec^2 x - 1\,dx = \tan x - x + C$$

6. Find $\int \cos^2 x \sin^3 x \, dx$

Solution: Odd power of $\sin x$, so substitute $u = \cos x$, $du = -\sin x \, dx$, $dx = -\frac{du}{\sin x}$,

$$\begin{aligned}
\int \cos^2 x \sin^3 x \, dx &= \int u^2 \sin^3 x \, \frac{-du}{\sin x} \\
&= \int -u^2 \sin^2 x \, du \\
&= \int -u^2(1-\cos^2 x) \, du \\
&= \int -u^2(1-u^2) \, du \\
&= \int u^4 - u^2 \, du \\
&= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\
&= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C
\end{aligned}$$

7. Find $\int \frac{x^2}{x+1} \, dx$

Solution: Not really a case for partial fractions, but covered by the general method for partial fractions. This is not a proper fraction. So do long division

$$\begin{array}{r|llll}
 & x & - & 1 & \\
\cline{2-5}
x+1 & x^2 & & & \\
 & x^2 & + & x & \\
\cline{2-4}
 & & - & x & \\
 & & - & x & - \quad 1 \\
\cline{4-5}
 & & & & \quad\; 1
\end{array}$$

Thus we have

$$\frac{x^2}{x+1} = \text{quotient} + \frac{\text{remainder}}{x+1} = x - 1 + \frac{1}{x+1}$$

$$\int \frac{x^2}{x+1} \, dx = \int \left(x - 1 + \frac{1}{x+1} \right) dx = \frac{1}{2}x^2 - x + \ln|x+1| + C$$

8. Find $\displaystyle\int_0^{\pi/6} e^{3x}\cos(2x)\,dx$

Solution: Integration by parts works here but it is tricky. Need to do it twice.

$$
\begin{array}{rcl}
\displaystyle\int_0^{\pi/6} e^{3x}\cos(2x)\,dx & & \text{Let} \quad \begin{array}{rclrcl} u &=& e^{3x} & dv &=& \cos(2x)\,dx \\ du &=& 3e^{3x}\,dx & v &=& \frac{1}{2}\sin(2x) \end{array} \\
\displaystyle\int_0^{\pi/6} e^{3x}\cos(2x)\,dx & = & \displaystyle\int_0^{\pi/6} u\,dv \\
& = & \displaystyle [uv]_0^{\pi/6} - \int_0^{\pi/6} v\,du \\
& = & \displaystyle \left[\frac{1}{2}e^{3x}\sin(2x)\right]_0^{\pi/6} - \int_0^{\pi/6}\frac{1}{2}\sin(2x)(3e^{3x})\,dx \\
& = & \displaystyle \frac{1}{2}e^{\pi/2}\sin(\pi/3) - \frac{1}{2}\sin 0 - \frac{3}{2}\int_0^{\pi/6} e^{3x}\sin(2x)\,dx \\
& & \text{Let} \quad \begin{array}{rclrcl} U &=& e^{3x} & dv &=& \sin(2x)\,dx \\ dU &=& 3e^{3x}\,dx & V &=& -\frac{1}{2}\cos(2x) \end{array} \\
& = & \displaystyle \frac{\sqrt{3}}{4}e^{\pi/2} - \frac{3}{2}\int_0^{\pi/6} U\,dV \\
& = & \displaystyle \frac{\sqrt{3}}{4}e^{\pi/2} - \frac{3}{2}\left([UV]_0^{\pi/6} - \int_0^{\pi/6} V\,dU\right) \\
& = & \displaystyle \frac{\sqrt{3}}{4}e^{\pi/2} - \frac{3}{2}\left(\left[-\frac{1}{2}e^{3x}\cos(2x)\right]_0^{\pi/6} - \int_0^{\pi/6}\left(-\frac{1}{2}\cos(2x)\right)(3e^{3x})\,dx\right) \\
& = & \displaystyle \frac{\sqrt{3}}{4}e^{\pi/2} + \frac{3}{4}[e^{3x}\cos(2x)]_0^{\pi/6} - \frac{9}{4}\int_0^{\pi/6} e^{3x}\cos(2x)\,dx \\
& = & \displaystyle \frac{\sqrt{3}}{4}e^{\pi/2} + \frac{3}{4}\left(e^{\pi/2}\left(\frac{1}{2}\right) - 1\right) - \frac{9}{4}\int_0^{\pi/6} e^{3x}\cos(2x)\,dx
\end{array}
$$

We appear to be back to where we started, which is looking for $\int_0^{\pi/6} e^{3x}\cos(2x)\,dx$, but we actually have an equation now that must be satisfied by this number.

Let $I = \int_0^{\pi/6} e^{3x}\cos(2x)\,dx$, and think of it as an unknown number. We have shown above that

$$I = \frac{\sqrt{3}}{4}e^{\pi/2} + \frac{3}{8}e^{\pi/2} - \frac{3}{4} - \frac{9}{4}I$$

That says

$$\left(1 + \frac{9}{4}\right)I = \frac{\sqrt{3}}{4}e^{\pi/2} + \frac{3}{8}e^{\pi/2} - \frac{3}{4}$$

and so

$$I = \frac{4}{13}\left(\frac{\sqrt{3}}{4}e^{\pi/2} + \frac{3}{8}e^{\pi/2} - \frac{3}{4}\right) = \frac{\sqrt{3}}{13}e^{\pi/2} + \frac{3}{26}e^{\pi/2} - \frac{3}{13}$$

9. Find $\displaystyle\int \frac{1}{x^2+x+1}\,dx$

   Solution: The denominator has complex roots (can't be factored without using complex numbers) and so we do this by completing the square

$$x^2+x+1 = x^2+x+\frac{1}{4}-\frac{1}{4}+1 = \left(x-\frac{1}{2}\right)^2+\frac{3}{4}$$

   It is possible to do a substitution $x-\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$ or to know that the answer is

$$\begin{aligned}\int \frac{1}{x^2+x+1}\,dx &= \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx \\ &= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x-1/2}{\sqrt{3}/2}\right)+C \\ &= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C\end{aligned}$$

10. Find $\displaystyle\int \frac{x^2}{(x+2)(x^2-1)}\,dx$

    Solution: Partial fractions. Proper fraction (denominator of degree 3, numerator of degree $2<3$). Factorise denominator completely $(x+2)(x^2-1) = (x+2)(x-1)(x+1)$. Partial fractions take the form

$$\frac{x^2}{(x+2)(x^2-1)} = \frac{x^2}{(x+2)(x-1)(x+1)} = \frac{A_1}{x+2}+\frac{A_2}{x-1}+\frac{A_3}{x+1}$$

    Can work out A_1, A_2 and A_3 by multiplying across by $(x+2)(x-1)(x+1)$ and then substituting $x=-2$, $x=1$ and $x=-1$ in turn.

    Result is

$$\begin{aligned}\int \frac{x^2}{(x+2)(x^2-1)}\,dx &= \int\left(\frac{4/3}{x+2}+\frac{1/6}{x-1}-\frac{1/2}{x+1}\right)dx \\ &= \frac{4}{3}\ln|x+2|+\frac{1}{6}\ln|x-1|-\frac{1}{2}\ln|x+1|+C\end{aligned}$$

11. Find $\displaystyle\int \frac{x^2}{(x+1)(x^2-1)}\,dx$

Solution: Partial fraction again. Again a proper fraction. Factorise denominator completely $(x+1)(x^2-1) = (x+1)(x-1)(x+1) = (x+1)^2(x-1)$. Partial fractions take the form

$$\frac{x^2}{(x+1)(x^2-1)} = \frac{x^2}{(x+1)^2(x-1)} = \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{A_3}{x-1}$$

Can work out A_1, A_2 and A_3 by multiplying across by $(x+1)^2(x-1)$ and then substituting $x=-1$, $x=1$ and $x=0$ in turn.

Result is

$$\begin{aligned}\int \frac{x^2}{(x+1)(x^2-1)}\,dx &= \int \left(\frac{3/4}{x+1} - \frac{1/2}{(x+1)^2} + \frac{1/4}{x-1}\right) dx \\ &= \frac{3}{4}\ln|x+1| + \frac{1}{2(x+1)} + \frac{1}{4}\ln|x-1| + C\end{aligned}$$

12. Find $\displaystyle\int \frac{x^2}{(x+1)(x^2+1)}\,dx$

Solution: Partial fraction again. Again a proper fraction. Denominator is already factorised completely. Partial fractions take the form

$$\frac{x^2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

Can work out A by multiplying across by $(x+1)(x^2+1)$ and then substituting $x=-1$. Substituting $x=0$ finds C and another x then gives B.

Result is

$$\begin{aligned}\int \frac{x^2}{(x+1)(x^2+1)}\,dx &= \int \left(\frac{1/2}{x+1} + \frac{(1/2)x - 1/2}{x^2+1}\right) dx \\ &= \frac{1}{2}\ln|x+1| + \int \left(\frac{x}{2(x^2+1)} - \frac{1}{2(x^2+1)}\right) dx \\ &= \frac{1}{2}\ln|x+1| + \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\tan^{-1}x + \text{ const}\end{aligned}$$

(I used '+ const' rather than the usual '$+C$' just because I used C already for something else.)

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