10.3.2 Integration by parts

If we integrate both sides of the product rule

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

we get

$$\int \frac{d}{dx}(uv)\,dx = \int v\frac{du}{dx}\,dx + \int u\frac{dv}{dx}\,dx$$

or

$$uv = \int v\frac{du}{dx}\,dx + \int u\frac{dv}{dx}\,dx$$

This allows us a way of transforming integrals that take the form of a product of one function times the derivative of another

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

into a different integral (where the differentiation has flipped from one factor to the other). The advantage of this comes if we know how to manage the new integral (or at least if it is simpler than the original). The integration by parts formula is usually written with the ds's cancelled

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Examples 10.3.2.1. (a) $\displaystyle\int x\ln x\,dx$

Solution: The two most obvious ways to use integration by parts are

- $u = x$, $dv = \ln x\,dx$ (Problem with this is we can't find v very easily)
- $u = \ln x$, $dv = x\,dx$

It turns out that the second is good.

$$\int x \ln x \, dx \qquad \text{Let} \quad \begin{array}{rclrcl} u & = & \ln x & dv & = & x\,dx \\ du & = & \frac{1}{x}\,dx & v & = & \frac{x^2}{2} \end{array}$$

$$\begin{aligned} \int x \ln x \, dx &= \int u \, dv \\ &= uv - \int v \, du \\ &= (\ln x)\frac{x^2}{2} - \int \frac{x^2}{2}\frac{1}{x}\,dx \\ &= (x^2/2)\ln x - \int \frac{x}{2}\,dx \\ &= (x^2/2)\ln x - \frac{x^2}{4} + C \end{aligned}$$

(b) $\displaystyle\int_1^e \ln x \, dx$

Solution: This is one of a very few cases which can be done by taking $dv = dx$ and $u =$ the integrand. Every integral takes the form $\int u\,dv$ in that way, but it is rarely a good way to start integration by parts.

Here we can make use of the definite integral form of the integration by parts formula

$$\boxed{\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du}$$

which arises in the same way as the indefinite integral formula (take definite integrals of the product rule for differentiation).

$$\int_1^e \ln x \, dx \qquad \text{Let} \quad \begin{array}{rclrcl} u & = & \ln x & dv & = & dx \\ du & = & \frac{1}{x}\,dx & v & = & x \end{array}$$

$$\begin{aligned} \int_1^e \ln x \, dx &= \int_1^e u \, dv \\ &= [uv]_1^e - \int_1^e v \, du \\ &= [(\ln x)x]_1^e - \int_1^e x\frac{1}{x}\,dx \\ &= e \ln e - \ln 1 - \int_1^e 1\,dx \\ &= e - [x]_1^e \\ &= e - (e-1) = 1 \end{aligned}$$

(c) $\int x^2 \cos x \, dx$

Solution:

$$\int x^2 \cos x \, dx \qquad \text{Let} \quad \begin{array}{rclrcl} u & = & x^2 & dv & = & \cos x \, dx \\ du & = & 2x \, dx & v & = & \sin x \end{array}$$

$$\begin{aligned} \int x^2 \cos x \, dx &= \int u \, dv \\ &= uv - \int v \, du \\ &= x^2 \sin x - \int \sin x (2x) \, dx \\ &= x^2 \sin x - \int 2x \sin x \, dx \end{aligned}$$

The point here is that we have succeeded in simplifying the problem. We started with x^2 times a trigonometric function ($\cos x$) and we have now got to x times a trigonometric function ($\sin x$ this time, but that is not so different in difficulty to $\cos x$). If we continue in the *same* (or similar) way and apply integration by parts again, we can make the problem even simpler. We use U and V this time in case we might get confused with the earlier u and v.[1]

$$\int 2x \sin x \, dx \qquad \text{Let} \quad \begin{array}{rclrcl} U & = & 2x & dV & = & \sin x \, dx \\ dU & = & 2 \, dx & V & = & -\cos x \end{array}$$

$$\begin{aligned} \int 2x \sin x \, dx &= \int U \, dV \\ &= UV - \int V \, dU \\ &= 2x(-\cos x) - \int (-\cos x) 2 \, dx \\ &= -2x \cos x + \int 2 \cos x \, dx \\ &= -2x \cos x + 2 \sin x + C \end{aligned}$$

Combining with the first stage of the calculation

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x - C$$

and, in fact $-C$ is plus another constant. Since C can be any constant, the answer

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C$$

is also good.

[1] One thing to avoid is $U = v$ and $V = u$ because this will just unravel what we did to begin with.

Remark 10.3.2.2. We will not in fact learn any other techniques than these which are purely integration methods. We will spend some time explaining how to make use of these techniques in specific circumstances (as it is often not at all obvious how to do so). There is one other method we will come to called *partial fractions*, a method for integrating fractions such as

$$\int \frac{x+2}{(x-1)(x^2+2x+2)}\,dx$$

However, the thing we have to learn about is algebra — a way to rewrite fractions like this as sums of simpler ones — and there is no new idea that is directly integration. The algebra allows us to tackle problems of this sort.

10.3.3 Trigonometric Integrals

(i) **Powers of $\sin x$ times powers of $\sin x$ with one power odd**

Method: For

$$\int \sin^n x \cos^m x\,dx$$

- if $n =$ the power of $\sin x$ is odd, substitute $u = \cos x$
- if $m =$ the power of $\cos x$ is odd, substitute $u = \sin x$

Example 10.3.3.1. $\displaystyle\int \sin^3 x \cos^4 x\,dx$

Solution: Let $u = \cos x$, $du = -\sin x\,dx$, $dx = \dfrac{du}{-\sin x}$

$$\begin{aligned}
\int \sin^3 x \cos^4 x\,dx &= \int \sin^3 x u^4 \frac{du}{-\sin x} \\
&= \int -\sin^2 x u^4\,du \\
&= \int -(1-\cos^2 x)u^4\,du \\
&= \int -(1-u^2)u^4\,du \\
&= \int u^6 - u^4\,du \\
&= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C \\
&= \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C
\end{aligned}$$

(ii) **Powers of $\sin x$ times powers of $\sin x$ with both powers even**

Method: use the trigonometric identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Example 10.3.3.2. $\displaystyle\int \sin^4 x \cos^2 x \, dx$

Solution:

$$\begin{aligned}
\int \sin^4 x \cos^2 x \, dx &= \int (\sin^2 x)^2 \cos^2 x \, dx \\
&= \int \left(\frac{1}{2}(1 - \cos 2x)\right)^2 \left(\frac{1}{2}(1 + \cos 2x)\right) \, dx \\
&= \frac{1}{8} \int (1 - 2\cos 2x + \cos^2 2x)(1 + \cos 2x) \, dx \\
&= \frac{1}{8} \int 1 - \cos 2x - \cos^2 2x + \cos^3 2x \, dx
\end{aligned}$$

Now $\int 1 \, dx$ is no bother. $\int \cos 2x \, dx$ is not much harder than $\int \cos x \, dx = \sin x + C$. If we look at

$$\frac{d}{dx} \sin 2x = (\cos 2x) 2$$

we can see that $\int \cos 2x \, dx = \frac{1}{2} \sin 2x + C$. (This can also be done by a substitution $u = 2x$ but that is hardly needed.) Next

$$\int \cos^2 2x \, dx = \int \frac{1}{2}(1 + \cos 4x) \, dx = \frac{1}{2}\left(x + \frac{1}{4} \sin 4x\right) + C$$

(using the same ideas as for $\int \cos 2x \, dx$).

For $\int \cos^3 2x \, dx$ we are in a situation where we have an odd power of cos times a zeroth power of sin. So we can use the earlier method (the fact that the angle is $2x$ doe snot make a big difference) of substituting $u = \sin 2x$. Then $du = 2 \cos 2x \, dx$,

$dx = \dfrac{du}{2\cos 2x}$,

$$\begin{aligned}
\int \cos^3 2x\,dx &= \int \cos^3 2x \frac{du}{2\cos 2x} \\
&= \frac{1}{2}\int \cos^2 2x\,du \\
&= \frac{1}{2}\int 1 - \sin^2 2x\,dx \\
&= \frac{1}{2}\int 1 - u^2\,du \\
&= \frac{1}{2}\left(u - \frac{1}{3}u^3\right) + C \\
&= \frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x + C
\end{aligned}$$

Putting all the bits together

$$\begin{aligned}
\int \sin^4 x\cos^2 x\,dx &= \frac{1}{8}\int 1 - \cos 2x - \cos^2 2x + \cos^3 2x\,dx \\
&= \frac{1}{8}\left(x - \frac{1}{2}\sin 2x - \frac{1}{2}x - \frac{1}{8}\sin 4x + \frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x\right) + C \\
&= \frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C
\end{aligned}$$

(iii) **Powers of $\sin x$ and $\cos x$**

Method: Use the previous two methods treating

$$\int \sin^n x\,dx = \int \sin^n x(\cos x)^0\,dx$$

and similarly for $\int \cos^m x\,dx$ (that is treat the second power as the zeroth power).

Examples 10.3.3.3. • $\int \cos^3 x\,dx$

Solution: $\int \cos^3 x\,dx = \int (\sin x)^0 \cos^3 x\,dx$. Power of cos odd. Substitute $u = \sin x$, $du = \cos x\,dx$, $dx = \dfrac{du}{\cos x}$

$$\begin{aligned}
\int \cos^3 x\,dx &= \int \cos^3 x \frac{du}{\cos x} \\
&= \int \cos^2 x\,du \\
&= \int 1 - u^2\,du \\
&= u - \frac{1}{3}u^3 + C \\
&= \sin x = \frac{1}{3}\sin^3 x + C
\end{aligned}$$

- $\int \sin^4 x\,dx$

  Solution: Use even powers method.

$$\begin{aligned}
\int \sin^4 x\,dx &= \int (\sin^2 x)^2\,dx \\
&= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2\,dx \\
&= \frac{1}{4}\int 1 - 2\cos 2x + \cos^2 2x\,dx \\
&\qquad \text{Note: still have one even power} \\
&= \frac{1}{4}\int 1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\,dx \\
&= \frac{1}{4}\int \frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\,dx \\
&= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + C \\
&= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C
\end{aligned}$$

10.3.4 Inverse trigonometric substitutions

We now consider a class of substitutions which seem quite counter intuitive.

Recall these

$$\begin{aligned}
\frac{d}{dx}\sin^{-1} x &= \frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}\tan^{-1} x &= \frac{1}{1+x^2} \\
\frac{d}{dx}\cosh^{-1} x &= \frac{1}{\sqrt{x^2-1}} \\
\frac{d}{dx}\sinh^{-1} x &= \frac{1}{\sqrt{x^2+1}}
\end{aligned}$$

The corresponding substitution methods are:

- integrals involving $\sqrt{1-x^2}$, substitute $\theta = \sin^{-1} x$ (or it is often more convenient to write it $x = \sin\theta$).

  More generally, integrals involving $\sqrt{a^2-u^2}$ (with a constant) — try substituting $u = a\sin\theta$ (or $\theta = \sin^{-1}(x/a)$).

- integrals involving $\dfrac{1}{a^2+u^2}$ — try substituting $u = a\tan\theta$

- integrals involving $\sqrt{u^2-a^2}$ — try substituting $u = a\cosh t$

- integrals involving $\sqrt{u^2 + a^2}$ — try substituting $u = a \sinh t$

Examples 10.3.4.1. (i) $\int \sqrt{9 - 4(x+2)^2}\, dx$

Solution: In this case, we have $\sqrt{a^2 - u^2}$ with $a^2 = 9$, $a = 3$, $u = 2(x+2)$ and so our method says to try $u = a \sin\theta$ or

$$\begin{aligned}
2(x+2) &= 3\sin\theta \\
2\, dx &= 3\cos\theta\, d\theta \\
dx &= \frac{3\cos\theta}{2}\, d\theta \\
\int \sqrt{9 - 4(x+2)^2}\, dx &= \int \sqrt{9 - 9\sin^2\theta}\frac{3\cos\theta}{2}\, d\theta \\
&= \int \sqrt{9\cos^2\theta}\frac{3\cos\theta}{2}\, d\theta \\
&= \int 3\cos\theta\frac{3\cos\theta}{2}\, d\theta \\
&= \frac{9}{2}\int \cos^2\theta\, d\theta \\
&= \frac{9}{4}\int 1 + \cos 2\theta\, d\theta \\
&= \frac{9}{4}(\theta + \frac{1}{2}\sin 2\theta) + C
\end{aligned}$$

To get the answer in terms of x, we need θ in terms of x

$$\begin{aligned}
2(x+2) &= 3\sin\theta \\
\frac{2}{3}(x+2) &= \sin\theta \\
\theta &= \sin^{-1}\left(\frac{2}{3}(x+2)\right)
\end{aligned}$$

and we could get a correct answer by replacing θ by this everywhere in the answer above.

There is a way to simplify the answer but we won't go into that.

(ii) $\int \sqrt{-4x^2 - 16x - 7}\, dx$

Solution: For quadratics inside a square root like this, what we should do first is complete the square. That is, rearrange the x^2 and x terms so that (with a suitable constant) we get a multiple of a perfect square

$$\begin{aligned} -4x^2 - 16x - 7 &= -4(x^2 + 4x) - 7 \\ &= -4(x^2 + 4x + 4 - 4) - 7 \\ &= -4(x^2 + 4x + 4) + 9 \\ &= 9 - 4(x + 2)^2 \end{aligned}$$

This means that not only is this problem similar to the previous one, it is in fact the same problem again (now that we completed the square).

Remark 10.3.4.2. There are in fact many more tricks we could go into.

10.3.5 Partial Fractions

Partial fractions are a technique from algebra, but our reason for dealing with them is that they can in principle help find every integral of the form

$$\int \frac{p(x)}{q(x)}\,dx$$

where $p(x)$ and $q(x)$ are polynomials.

Except in a few special cases, we don't yet know how to find such integrals. One special case, where we don't need partial fractions, is where $q'(x) = p(x)$ or $q'(x) = \alpha p(x)$ for some constant α, because in these cases we can make a substitution $u = q(x)$, $du = q'(x)\,dx$ and it will work out nicely. In fact substitution would also work if $q(x) = r(x)^n$ for some $n \geq 1$ and $r'(x) = \alpha p(x)$ for a constant α — we can substitute $u = r(x)$, $du = r'(x)\,dx = \alpha p(x)\,dx$,

$$\int \frac{p(x)}{q(x)}\,dx = \int \frac{p(x)}{r(x)^n}\,dx = \int \frac{p(x)}{u^n}\,\frac{du}{\alpha p(x)} = \int (1/\alpha)\frac{1}{u^n}\,du$$

The idea of partial fractions is to rewrite $\frac{p(x)}{q(x)}$ as a sum of fractions with simple denominators and numerators that are somehow small compared to the denominator. We need to explain exactly how it goes.

We need to talk about factoring polynomials as much as possible.

To start with, a polynomial is an expression you get by taking a finite number of powers of x with constant coefficients in front and adding them up. For example

$$p(x) = 4x^2 - x + 17$$

or

$$p(x) = 27x^{11} + 15x^{10} - x^9 + x^8 + 11x^2 + 5$$

are polynomials. The highest power of x that has a nonzero coefficient in front is called the *degree* of the polynomial. The examples above have degree 2 and degree 11.

What is handy to know is that when we multiply polynomials, the degrees add. So $(x+1)(x+5)(x^2+x+11)$ will have degree $1+1+2=4$ when it is multiplied out. Constant polynomials have degree 0, except the zero polynomial — we are best not giving any degree to the zero polynomial.

Now, what kind of polynomial can be factored? For this purpose we don't consider constant factors as genuine factors. So

$$2x^2 + 4 = 2(x^2 + 2) = \frac{1}{3}(6x^2 + 12)$$

will not be counted as factorisations.

Anything of degree 1 certainly cannot be factored then. Some things of degree 2 can be factored, such as

$$x^2 + 5x + 4 = (x + 1)(x + 4)$$

but other quadratics cannot be factored if we don't allow complex numbers to be used. We *cannot factor*

$$x^2 + 2x + 2 = (x - \alpha)(x - \beta)$$

because if we could then the roots of x^2+2x+2 would be α and β. The roots of x^2+2x+2 are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm \sqrt{-1}$$

and these are complex numbers. So α and β would have to be these complex numbers.

A remarkable fact is that every polynomial of degree 3 or more **can** be factored, at least in theory. It does not mean it is easy to find the factors, unfortunately. What you can sometimes rely on to factor polynomials is the **Remainder Theorem.** Recall that it says that if $p(x)$ is a polynomial and you know a root $x = a$ (that is a value a so that $p(a) = 0$), then $x - a$ must divide $p(x)$.

Using the theory, just as in principle whole numbers can be factored as a product of prime numbers, so polynomials $p(x)$ with real coefficients can be factored as a product of linear factors like $x - a$ and quadratic factors $x^2 + bx + c$ with complex roots. If the coefficient of the highest power of x in $p(x)$ is not 1, then we also need to include that coefficient. So a complete factorisation of

$$2x^2 + 8x + 2 = (2x + 4)(x + 2) = 2(x + 2)(x + 2) = 2(x + 2)^2.$$

For $3x^3 + 3x^2 + 6x + 6 = 3(x^3 + x^2 + 2x + 2)$, you can check that $x = -1$ is a root and so $x - (-1) = x + 1$ must divide it. We get

$$3x^3 + 3x^2 + 6x + 6 = 3(x + 1)(x^2 + 2)$$

from long division.

Now we can outline **how partial fractions works** for a fraction $\frac{p(x)}{q(x)}$ of two polynomials:

Step 0: (preparatory step). If the degree of the numerator $p(x)$ is not strictly smaller than the degree of the denominator $q(x)$, use long division to divide $q(x)$ into $p(x)$ and obtain a quotient $Q(x)$ and remainder $R(x)$. Then

$$\frac{p(x)}{q(x)} = Q(x) + \frac{R(x)}{q(x)}$$

and $\text{degree}(R(x)) < \text{degree}(q(x))$.

We concentrate then on the 'proper fraction' part $R(x)/q(x)$.

Step 1: Now factor $q(x)$ *completely* into a product of linear factors $x - a$ and quadratic factors $x^2 + bx + c$ with complex roots.

Gather up any repeated terms, so that if (say) $q(x) = (x-1)(x+2)(x^2+3)(x+2)$ we would write it as $q(x) = (x-1)(x+2)^2(x^2+3)$.

Step 2: Then the proper fraction $\dfrac{R(x)}{q(x)}$ can be written as a sum of fractions of the following types:

(i) $\dfrac{A}{(x-a)^m}$

(ii) $\dfrac{Bx + C}{(x^2 + bx + c)^k}$

where we include all possible powers $(x-a)^m$ and $(x^2+bx+c)^k$ that divide $q(x)$. The A, B, C stand for constants.

As examples, consider the following. We just write down what the partial fractions look like. In each case, we start with a proper fraction where the denominator is completely factored already. So some of the hard work is already done.

(i) $\dfrac{x^2+x+5}{(x-1)(x-2)(x-3)} = \dfrac{A_1}{x-1} + \dfrac{A_2}{x-2} + \dfrac{A_3}{x-3}$

(ii) $\dfrac{x^3+2x+7}{(x+1)^2(x-4)} = \dfrac{A_1}{x+1} + \dfrac{A_2}{(x+1)^2} + \dfrac{A_3}{x-4}$

(iii) $\dfrac{x^2+x+11}{(x+1)(x^2+2x+2)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+2x+2}$

(iv) $\dfrac{x^4+x+11}{(x+1)(x^2+2x+2)^2} = \dfrac{A}{x+1} + \dfrac{B_1x+C_1}{x^2+2x+2} + \dfrac{B_2x+C_2}{(x^2+2x+2)^2}$

To make use of these, we have to be able to find the numbers $A, B, C, \ldots$ that make the equation true. Take the first example

$$\frac{x^2+x+5}{(x-1)(x-2)(x-3)} = \frac{A_1}{x-1} + \frac{A_2}{x-2} + \frac{A_3}{x-3}.$$

To find the appropriate A_1, A_2, A_3, we multiply across by the original denominator $(x-1)(x-2)(x-3)$. This has the effect of clearing all the fractions.

$$\begin{aligned}
x^2+x+5 &= \frac{A_1}{x-1}(x-1)(x-2)(x-3) + \frac{A_2}{x-2}(x-1)(x-2)(x-3) \\
&\quad + \frac{A_3}{x-3}(x-1)(x-2)(x-3) \\
&= A_1(x-2)(x-3) + A_2(x-1)(x-3) + A_3(x-1)(x-2)
\end{aligned}$$

There are two avenues to pursue from here. In this case, method 1 seems easier to me, but in general method 2 can be as good.

Method 1. Plug in the values of x that make the original denominator $(x-1)(x-2)(x-3) = 0$.

$$\begin{aligned}
x=1 &: \\
1+1+5 &= A_1(1-2)(1-3) + A_2(0) + A_3(0) \\
7 &= 2A_1 \\
& \boxed{A_1 = 7/2} \\
x=2 &: \\
11 &= A_1(0) + A_2(1)(-1) + A_3(0) \\
&= -A_2 \\
& \boxed{A_2 = -11} \\
x=3 &: \\
17 &= 0+0+A_3(2)(1) \\
& \boxed{A_3 = 17/2}
\end{aligned}$$

So we get

$$\frac{x^2+x+5}{(x-1)(x-2)(x-3)} = \frac{7/2}{x-1} + \frac{-11}{x-2} + \frac{17/2}{x-3}.$$

Our interest in this is for integration. We can now easily integrate

$$\begin{aligned}
\int \frac{x^2+x+5}{(x-1)(x-2)(x-3)}\,dx &= \int \frac{7/2}{x-1} + \frac{-11}{x-2} + \frac{17/2}{x-3}\,dx \\
&= \frac{7}{2}\ln|x-1| - 11\ln|x-2| + \frac{17}{2}\ln|x-3| + C
\end{aligned}$$

Method 2. Multiply out the right hand side.

$$\begin{aligned}
x^2+x+5 &= A_1(x-2)(x-3) + A_2(x-1)(x-3) + A_3(x-1)(x-2) \\
&= A_1(x^2-5x+6) + A_2(x^2-4x+3) + A_3(x^2-2x+2) \\
&= (A_1+A_2+A_3)x^2 + (-5A_1-4A_2-2A_3)x + (6A_1+3A_2+2A_2)
\end{aligned}$$

and compare the coefficients on both sides to get a system of linear equations

$$\begin{array}{rcrcrcl} A_1 & + & A_2 & + & A_3 & = & 1 \\ -5A_1 & - & 4A_2 & - & 2A_3 & = & 1 \\ 6A_1 & + & 3A_2 & + & 2A_2 & = & 5 \end{array}$$

These can be solved (by Gaussian elimination, for example) to find A_1, A_2, A_3.

Method 1 is certainly magic in this case, but there are examples where Method 1 does not get all the unknown so easily.

Another example. Here is one of our previous examples with the numbers worked out.

$$\frac{x^2+x+11}{(x+1)(x^2+2x+2)} = \frac{11}{x+1} + \frac{-10x-11}{x^2+2x+2}$$

To find the integral of this,

$$\begin{aligned} \int \frac{x^2+x+11}{(x+1)(x^2+2x+2)}\,dx &= \int \frac{11}{x+1}\,dx - \int \frac{10x+11}{x^2+2x+2}\,dx \\ &= 11\ln|x+1| - \int \frac{10x+11}{x^2+2x+2}\,dx \end{aligned}$$

To work out the remaining integral, we use the method of completing the square $x^2+2x+2 = x^2+2x+1+1 = (x+1)^2+1$ and there is a trick. The trick is inspired by the fact that the substitution $u = x^2+2x+2$, $du = (2x+2)\,dx = 2(x+1)\,dx$ would work if the numerator was a multiple of $x+1$. What we can do is write

$$\int \frac{10x+11}{x^2+2x+2}\,dx = \int \frac{10x+10}{x^2+2x+2}\,dx + \int \frac{1}{x^2+2x+2}\,dx$$

and make the $u = x^2+2x+2$ substitution in the first half, while the second is an inverse tan example. (By substituting $w = x+1$, $dw = dx$, the second part becomes

$$\int \frac{1}{(x+1)^2+1}\,dx = \int \frac{1}{w^2+1}\,dw = \tan^{-1} w = \tan^{-1}(x+1)$$

or we might be able to guess that.) We get

$$\begin{aligned} \int \frac{10x+11}{x^2+2x+2}\,dx &= \int \frac{10x+10}{u}\,\frac{du}{2(x+1)} + \int \frac{1}{(x+1)^2+1}\,dx \\ &= \int \frac{5}{u}\,du + \tan^{-1}(x+1) \\ &= 5\ln|u| + \tan^{-1}(x+1) + C \\ &= 5\ln(x^2+2x+2) + \tan^{-1}(x+1) + C \end{aligned}$$

Finally, our integral works out as

$$\int \frac{x^2+x+11}{(x+1)(x^2+2x+2)}\,dx = 11\ln|x+1| - 5\ln(x^2+2x+2) - \tan^{-1}(x+1) + C$$