As in the rate of change explanation of the derivative for scalar-valued functions, we can argue in the case of vector-valued $\mathbf{x}(t)$ that $\mathbf{x}'(t)$ is a velocity vector, and that it should be a tangent vector to the curve.

Leaving aside the motivation, we take it as a definition that if $\mathbf{x}'(t_0)$ is not zero, then it is the tangent to the parametric curve. We define the line through the point $\mathbf{x}(t_0)$ in the direction $\mathbf{x}'(t_0)$ to be the tangent line to the curve.

One thing that makes this definition satisfactory is that if we drive along the same curve at a different rate, we get the same tangent direction. To explain that a bit better, suppose $t = t(s)$ is a monotone increasing function with $dt/ds > 0$ always. Then $\mathbf{y}(s) = \mathbf{x}(t(s))$ defines a new parametric curve, but it traces out the same points as $\mathbf{x}(t)$ (even in the same order). From the ordinary chain rule applied to each of the components we can conclude that

$$\frac{d\mathbf{y}}{ds} = \frac{d\mathbf{x}}{dt}\frac{dt}{ds}$$

and so $\mathbf{y}'$ has the same direction as $\mathbf{x}'$.

9.4 Functions of two variables

We recall now that a function $f\colon A \to B$ from a set A to a set B is a *rule* that assigns one and only one value $f(a)$ in B to each element $a \in A$. So far we have had $A \subseteq \mathbb{R}$. First we also had $B \subseteq \mathbb{R}$ but in the subsection just above we had $B = \mathbb{R}^n$ (or $B \subseteq \mathbb{R}^n$).

Our goal now is to consider the situation $A \subseteq \mathbb{R}^2$ and $B \subseteq \mathbb{R}$. So a function $f\colon A \to B$ now has a scalar value $f(a)$ at each point $a \in A$. Expanding a bit, we have a scalar value $f(x, y)$ at each point $(x, y) \in A \subseteq \mathbb{R}^2$. This explains the terminology 'function of two variables'.

A little later we will probably switch to $f(x_1, x_2)$ rather than $f(x, y)$. That is more economical on letters and fits better with generalising formulae to more variables. For now, we will keep to $f(x, y)$.

So, examples can come from formulae, such as

$$f(x, y) = \sqrt{x^2 + y^2} = \|(x, y)\|$$

or

$$f(x, y) = x^2 \cos^2 y + y^2 \cos^2 x$$

or

$$f(x, y) = \frac{1}{1 - (x^2 + y^2)}.$$

In that last example, the largest domain we could allow would be $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \neq 1\}$ (the complementary set to the unit circle).

9.4.1 Graphs of functions of two variables

For functions of one variable (scalar valued), we used the graph as an important way to picture the function. Notice that we needed a second dimension to fit in the graph. For functions $f(x, y)$ of two variables, we can also have a graph, but it needs $\mathbb{R}^3$ to fit it.

We already need a plane (think of the horizontal plane or the floor) to picture the points (x, y) in the domain of the function, and then what we do for the graph is plot one point (x, y, z) with $z = f(x, y)$ the height or altitude of the point above (x, y) in the horizontal plane.

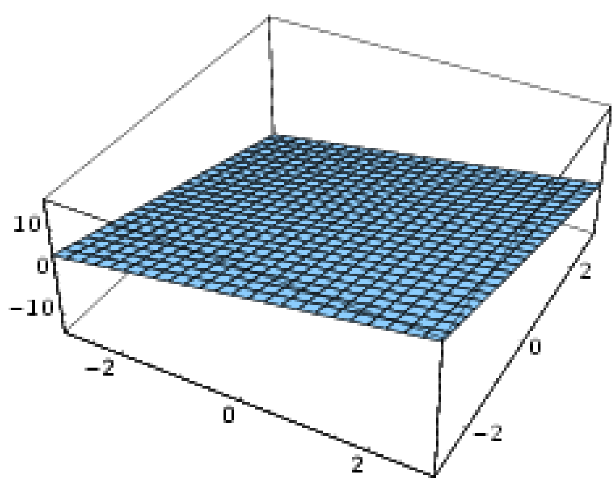
Examples 9.4.1. (i) For example in the case $f(x, y) = 1 + 2x - 3y$ the graph $z = f(x, y)$ is

$$z = 1 + 2x - 3y$$

and we can recognise that as the equation of a plane

$$-2x + 3y + z = 1$$

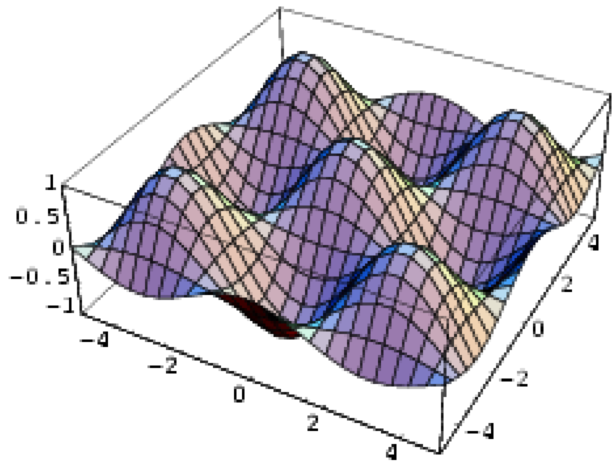
which has $(-2, 3, 1) = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ as a normal vector. Perhaps a way to draw it is to notice that the plane crosses the axes in $(-1/2, 0, 0)$, $(0, 1/3, 0)$ and $(0, 0, 1)$. (There is just one plane through 3 points, unless the points are in a line.) Here is a plot of that plane drawn by the program Mathematica:



(ii) In the case $f(x, y) = \cos x \cos y$, we can try to piece together what the graph $z = f(x, y)$ looks like by looking at sections of the graph where x or y is constant. For example if we freeze $y = 0$, we are looking at the section of the whole graph above the x-axis, where we see $z = f(x, 0) = \cos x$, a standard cosine wave starting at $z = 1$ when $x = 0$.

If we look at what happens when $x = 0$, which is along the y-axis, we get $z = f(0, y) = \cos y$ — the same shape again. If, instead of taking a section with $x = 0$ we take a section where $x = x_0$ is some other constant we get the graph $z = \cos x_0 \cos y$. This is a modified version of the regular cosine wave $z = \cos y$, which now oscillates between $z = \cos x_0$ and $z = -\cos x_0$. Looking at sections in the perpendicular direction, where $y = y_0$ is constant we get a similar shape $z = f(x, y_0) = \cos x \cos y_0 = (\cos y_0) \cos x$.

The whole graph $z = f(x, y)$ is a sort of landscape. If we think of the x-axis as pointing East and the y-axis as pointing North (along the horizontal floor), we have a landscape that always looks like a cosine wave if we follow it either East-West or North-South. Perhaps it takes some intuition to see what it is really like and so here is a picture drawn by Mathematica.



You can see peaks and valleys. The highest points happen when $\cos x = \sin y = 1$, or when $\cos x = \sin y = -1$, and there we get $z = 1$. The lowest points happen when $\cos x = 1$ and $\sin y = -1$, or when $\cos x = -1$ and $\sin y = -1$. At these lowest points we get $z = -1$.

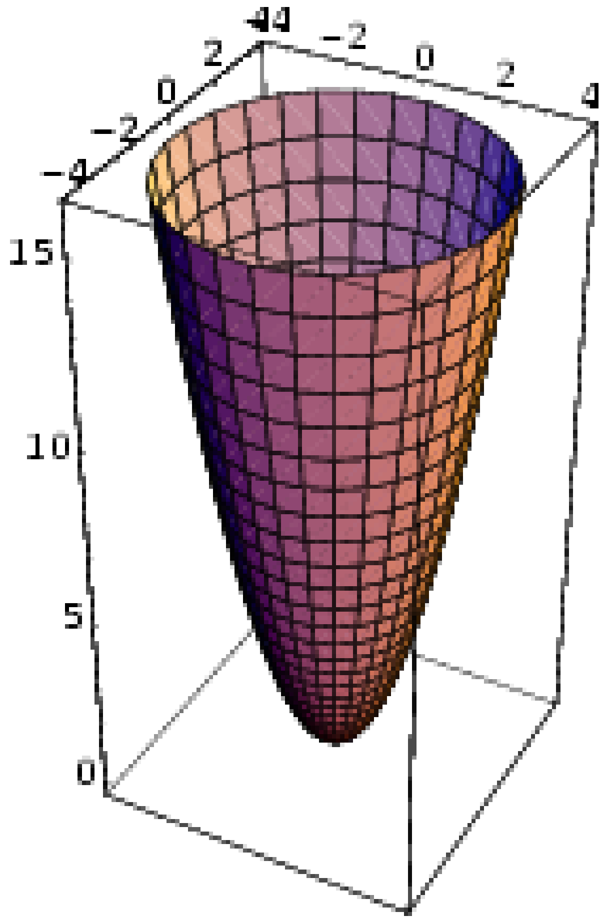
This may be a good place to explain that the altitude $z = f(x, y)$ of the points on a graph can be negative as well as positive. Negative altitude points happen when the graph dips below the floor level.

(iii) Next we look at $z = x^2 + y^2$, a somewhat easier example to figure out. This is the graph for $f(x, y) = x^2 + y^2$, and if you look to see what happens when $y = 0$ we get $z = f(x, 0) = x^2 + 0$, or $z = x^2$. This is a familiar parabola graph of x^2. For the section along the y-axis, where $x = 0$ we get $z = f(0, y) = y^2$ and this is again the same shape.

If you take a section of this graph in some other direction from the origin, you get still the same shape because

$$f(x, y) = x^2 + y^2 = (\text{dist}((x, y), (0, 0)))^2.$$

When you piece this information together, you can see that this is what the graph looks like.



This is a graph that is rotationally symmetric around the z-axis and looks like a standard parabola (x^2 graph) when you take any vertical section through the origin. It is called a paraboloid and in fact this is the shape for a reflector in a standard torch (or car head-lamps when they used to be just round). So it is a shape that has some applications.

9.4.2 Partial derivatives

We move on now to calculus for functions of two variables. There are a number of different concepts to absorb, but the simplest to explain is the idea of a partial derivative.

Start with a function $z = f(x, y)$ (which we can visualise in terms of its graph) and a point (x_0, y_0). To get the partial derivative with respect to x of $f(x, y)$ at the point (x_0, y_0) we consider the function of a single variable x that we get by fixing y to be y_0. That is the function $x \mapsto f(x, y_0)$, or $z = f(x, y_0)$.