8.6 Inverse hyperbolic functions

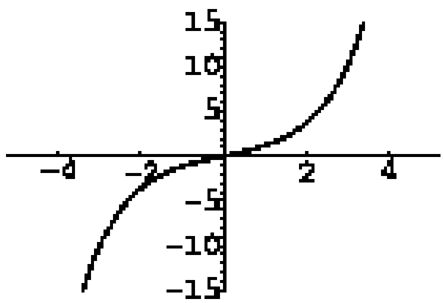
The hyperbolic sine function $y = \sinh x$ is strictly monotone increasing because $\frac{d}{dx} \sinh x = \cosh x = (e^x + e^{-x})/2 > 0$ always. So it has an inverse function. For $x > 0$ we have $\sinh x = (e^x - e^{-x})/2 > (e^x - 1)/2$ (since $e^{-x} < e^0 = 1$ for $x > 0$, or $-x < 0$). This shows that $\lim_{x\to\infty} \sinh x = \infty$. As $\sinh(-x) = -\sinh x$, we also have $\lim_{x\to-\infty} \sinh x = -\infty$.

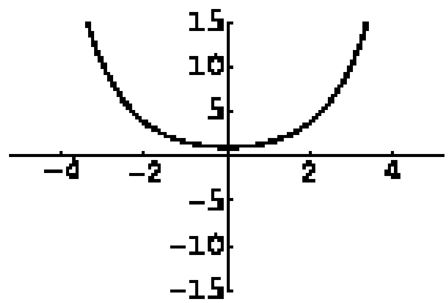
The hyperbolic cosine function $y = \cosh x$ is always positive. In fact $\cosh x \geq \cos 0 = 1$. We have $\frac{d}{dx} \cosh x = \sinh x$. For $x > 0$, we have $\sinh x > \sinh 0 = 0$ and so $\cosh x$ is strictly monotone increasing for $x > 0$. On the other hand, for $x < 0$, $\sinh x < \sinh 0$ and so $\cosh x$ is strictly monotone decreasing for $x < 0$. Also we have

$$\cosh x > \frac{1}{2} \max(e^x, e^{-x}) = \frac{1}{2} e^{|x|}$$

so that $\lim_{x\to\infty} \cosh x = \infty$ and also $\lim_{x\to-\infty} \cosh x = \infty$. In fact $\cosh x$ grows very rapidly, comparably fast to the exponential.

Here are graphs of $y = \sinh x$ and $y = \cosh x$.





The function $y = \sinh x$ has an inverse function $\sinh^{-1} \colon \mathbb{R} \to \mathbb{R}$. We can say then that

$$y = \sinh^{-1} x \text{ means exactly the same as } \sinh y = x$$

and the graph of $y = \sinh^{-1} x$ is the reflection of the graph of sinh in the line $y = x$. We can find dy/dx for $y = \sinh^{-1} x$ by the theorem on derivatives of inverse functions:

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{\cosh y}$$

and we can expresse that in terms of x using $\cosh^2 y - \sinh^2 y = 1$, $\cosh^2 y = 1 + \sinh^2 y = 1 + x^2$, to get

$$\frac{dy}{dx} = \boxed{\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1 + x^2}}}$$

(and it is right to have the square root because $\cosh y > 0$ always).

Proposition 8.6.1. *We can express* $\sinh^{-1} x$ *in terms of the natural logarithm as*

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

Proof. We can solve $\sinh y = x$ for y in terms of x as folows.

$$\begin{aligned} \frac{e^y - e^{-y}}{2} &= x \\ e^y - e^{-y} - 2x &= 0 \\ (e^y)^2 - 2xe^y - 1 &= 0 \end{aligned}$$

This is a quadradic equation for e^y. We get

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Since $\sqrt{x^2+1} > \sqrt{x^2} = |x|$, the minus sign would certainly make e^y negative — which is not possible. So we must have $e^y = x + \sqrt{x^2+1}$, and that means $y = \ln(x + \sqrt{x^2+1})$ as we required. □

For $\cosh^{-1}$, we have to interpret what we mean by the inverse because horizontal lines can cross the graph of cosh more than once.

Definition 8.6.2. By $\cosh^{-1}$ we mean the function $\cosh^{-1} : [1, \infty) \to [0, \infty)$ given by the rule

$$y = \cosh^{-1} x \text{ means } \cosh y = x \text{ and } y \geq 0.$$

We can find the derivative (as long as we don't go to the end point $x = 1$) in a similar way to the way we did above for $\sinh^{-1} x$. It is

$$\boxed{\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}} \; (x > 1).$$

We can also express $\cosh^{-1}$ via the natural logarithm as

$$\boxed{\cosh^{-1} x = \ln(x + \sqrt{x^2-1})} \; (x \geq 1).$$

Remark 8.6.3. There is an inverse for $y = \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. It is given by

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x} \qquad (-1 < x < 1),$$

and it has derivative

$$\frac{d}{dx} \tanh^{-1} x = \frac{1}{1-x^2} \qquad (-1 < x < 1).$$

Remark 8.6.4. It may be of interest to know that the graph of $y = \cosh x$ has the shape of a 'catenary', meaning the shape of a hanging chain (undisturbed by wind). By a 'chain' is meant something like a cable, but one that bends without reistance, yet still has mass.