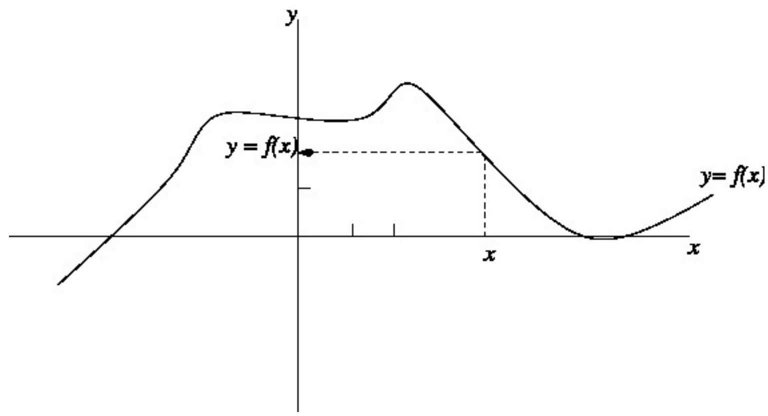
8 Inverse functions

8.1 General remarks

In this context it is useful to think of a function $y = f(x)$ as a machine (or black box) that does something predictable to an input x to produce an output y. Recall the diagram



from §1.5.

The inverse function is supposed to undo what the function does (or do the reverse). So the inverse function should send y back to x, if the function sends x to y. (Think of reversing the arrows.)

You may also realise that the q^{th} root function $y \mapsto y^{1/q}$ is a kind of inverse of the q^{th} power function $x \mapsto x^q$ (or $y = x^q$). If you recall from the discussion earlier, this was no problem for odd q, but it was a problem for even q. The problem was that many values of y came from more than one x and so it is not clear how to choose which x to send such a y back to.

What we need in order to have an inverse function is that no horizontal line crosses the graph more than once. So the picture above does not have an inverse function and neither does $y = x^2$ (or $y = x^4$ or $y = x^6$) have a genuine inverse. When we define the square root function to be the positive square root, we are (rather arbitrarily) deciding to cut down the function $y = x^2$ to the domain $x \geq 0$. The cut-down function then has an inverse.

Here is a handy fact.

Proposition 8.1.1. *If $A \subset \mathbb{R}$ and $f \colon A \to \mathbb{R}$ is a strictly monotone function, then no horizontal line crosses the graph $y = f(x)$ more than once.*

Proof. Let's take the case where f is strictly monotone increasing. Say there are two points on the graph $y = f(x)$ which are also on the horizontal line $y = y_0$. That means there are two different points $x_1, x_2 \in A$ with $(x_1, f(x_1)) = (x_1, y_0)$ and $(x_2, f(x_2)) = (x_2, y_0)$. Since $x_1 \neq x_2$ we must have either $x_1 < x_2$ or $x_2 < x_1$. If $x_1 < x_2$, then $f(x_1) < f(x_2)$ by definition of what it means for f to be strictly monotone increasing. But that is not so as $f(x_1) = f(x_2) = y_0$. On the other hand if $x_2 < x_1$, then $f(x_2) < f(x_1)$ (which is again not so).

For the case where f is strictly monotone decreasing, we can repeat almost the same argument, with small changes at the end. $\square$

There is in fact a theorem that says that the converse of the proposition is true for certain nice functions on intervals (called continuous functions — differentiable functions are included).

We concentrate then on strictly monotone functions $f\colon A \to \mathbb{R}$ with A an interval. Usually the range $B = \{f(x) : x \in A\}$ will also be an interval. (Technically there is a theorem that if $f(x)$ is continuous on the interval A, then B will be an interval.) The inverse function will be the function $g(y)$ with domain B given by the rule

$$g(y) = x \text{ exactly when } y = f(x).$$

It is usual to write f^{-1} rather than g for the inverse function. So we get

$$f^{-1}(y) = x \text{ exactly when } y = f(x).$$

Theorem 8.1.2. *If $y = f(x)$ is a function defined on an interval A and if $f'(x) > 0$ for all $x \in A$ then $x = f^{-1}(y)$ is differentiable on its domain and has derivative*

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}.$$

If instead $f'(x) < 0$ for all $x \in A$, then the same conclusion holds.

This rule is perhaps easier to remember as

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{\left(\frac{dy}{dx}\right)}.$$

We will not prove the theorem as it is quite tricky to prove it correctly. From implicit differentiation is is quite easy to see that that formula for dx/dy has to be right **IF** we already know that there is a derivative dx/dy for the inverse function.

Exercise 8.1.3. Check that the theorem gives the right result for the case of the function $y = x^q$ where we restrict $x > 0$ (and $q \in \mathbb{N}$).

8.2 The natural logarithm

The exponential function $y = e^x$ has strictly positive derivative for all $x \in (-\infty, \infty) = \mathbb{R}$. So it will have an inverse function. The domain of the inverse function will be $(0, \infty)$ according to our earlier considerations of the exponential (and its graph).

Definition 8.2.1. The natural logarithm function $\ln\colon (0, \infty) \to \mathbb{R}$ is the inverse of the exponential function.

Remark 8.2.2. So if we start with $y = e^x$ then

$$\ln y = x \text{ means exactly that } y = e^x.$$

Exchanging the roles of x and y we get

$$\ln x = y \text{ means exactly that } x = e^y.$$

Note that $\ln x$ only makes sense for $x > 0$.

Properties of the natural logarithm

(i) $\ln e = 1$ (because $e = e^1$).

(ii) $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$ (for $x > 0$).

Proof. Write $y = \ln x$, so that $x = e^y$. That tells us

$$\frac{dx}{dy} = e^y$$

and so

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}. \qquad \square$$

(iii) For $a > 0$, $\ln(1/a) = -\ln a$.

Proof. Let $x = \ln a$ so that $e^x = a$. But then we know $e^{-x} = 1/e^x = 1/a$. So $-x = \ln(1/a)$. That says $-\ln a = \ln(1/a)$. $\square$

(iv) For $a, b > 0$, $\ln(ab) = \ln a + \ln b$.

Proof. Let $x = \ln a$ and $t = \ln b$. Then we know $e^x = a$ and $e^t = b$. From that we have $ab = e^x e^t = e^{x+t}$ and then

$$\ln(ab) = x + t = \ln a + \ln b. \qquad \square$$

(v) For $x > 0$ and $r \in \mathbb{Q}$, $\ln(x^r) = r \ln x$.

Proof. Put $y = \ln x$ so that $e^y = x$. Then we know $(e^y)^r = e^{ry}$ (from properties of the exponential) and so $x^r = e^{ry}$ — which means exactly that $\ln(x^r) = ry = r \ln x$. $\square$

(vi) The graph of the natural logarithm function is obtained by reflecting the graph of the exponential in the diagonal line $y = x$. That is because the graph of $y = \ln x$ is the same as the graph of $e^y = x$.

8.3 Arbitrary powers of positive numbers

We were able to prove above that $\ln(x^r) = r \ln x$ for $x > 0$ and $r \in \mathbb{Q}$. Why did we not prove it for $r \in \mathbb{R}$?

The answer is that we have not yet defined x^a for $x > 0$ and $a \in \mathbb{R}$. We did define $e^a = \exp(a)$, and in that we thought we were on safe ground because $e^r = \exp(r)$ for $r \in \mathbb{Q}$.

For $x > 0$ we do know

$$x = e^{\ln x}$$

(because that is what inverse functions do — one undoes what the other does). So for $r \in \mathbb{Q}$ we do know

$$x^r = \left(e^{\ln x}\right)^r = e^{r \ln x}.$$

Taking this as a guide we can make it into a definition for arbitrary powers, not just rational powers.

Definition 8.3.1. For $x > 0$ and $a \in \mathbb{R}$ we define

$$x^a = e^{a \ln x} \qquad \text{(which may be easier to remember as } x^a = (e^{\ln x})^a = e^{a \ln x}\text{)}.$$

Proposition 8.3.2. *With this definition the laws of exponents hold: For $x, y > 0$ and $a, b \in \mathbb{R}$ we have*

(i) $1/(x^a) = x^{-a}$

(ii) $(x^a)^b = x^{ab}$

(iii) $x^a x^b = x^{a+b}$

(iv) $(xy)^a = x^a y^a$

This is not hard to prove based on the above definition and the properties of the exponential that we already know.

Examples 8.3.3. i) For $x > 0$ and $a \in \mathbb{R}$

$$\frac{d}{dx} x^a = \frac{d}{dx} \left(e^{\ln x}\right)^a = \frac{d}{dx} e^{a \ln x} = e^{a \ln x} a \frac{1}{x} = a x^a x^{-1} = a x^{a-1}.$$

ii) For $a > 0$ and $x \in \mathbb{R}$

$$\frac{d}{dx} a^x = \frac{d}{dx} \left(e^{\ln a}\right)^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \ln a = (\ln a) a^x.$$

Exercise 8.3.4. Find $\dfrac{d}{dx} x^x$.

8.4 Inverse trigonometric functions

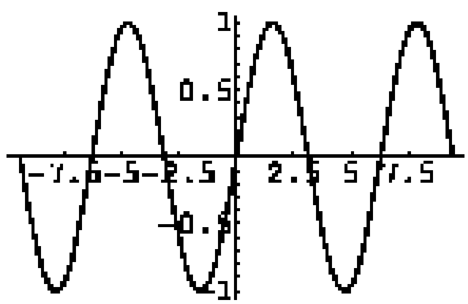
The trigonometric functions $y = \sin x$, $y = \cos x$, $y = \tan x$ (and so on) do not have any inverse in the ordinary way because horizontal lines often cut their graphs many times. In fact they are all periodic with period 2π (and $\tan x$ has period π), which means that

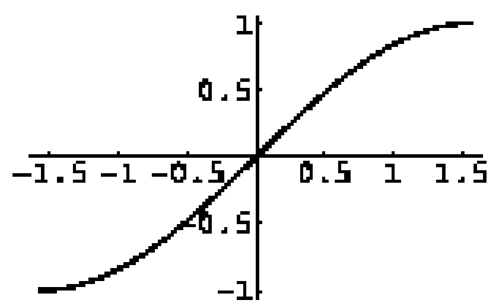
$$\sin(x + 2\pi) = \sin x, \quad \cos(x + 2\pi) = \cos x, \quad \tan(x + \pi) = \tan x.$$

So if we know $y = \sin x$, there is no way to know what x is, no inverse function.

We've seen this before. The function $y = x^2$ has no inverse either, but we came up with $\sqrt{y}$ by taking the positive square root. That is probably unfair discrimination against negative numbers, but it seems handy to have a square root function.

We do even more drastic things to come up with 'inverse' trigonometric functions. Here is the graph of $y = \sin x$ (for $-3\pi \le x \le 3\pi$) and a graph of a very much restricted $y = \sin x$, restricted to $-\pi/2 \le x \le \pi/2$.





The restricted graph is a strictly monotone increasing graph and when we write $\sin^{-1} y$ we mean the inverse function of this cut down $y = \sin x$. (Some people use the notation $\arcsin y$ instead of $\sin^{-1} y$. That emphasises that it is not really an inverse function.)

In summary

$$x = \sin^{-1} y \text{ means exactly that } \sin x = y \text{ and } -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

Changing around the names of the variables, we get

Definition 8.4.1. The '*inverse* sin *function*' (or arcsin function) is the function $\sin^{-1} \colon [-1, 1] \to [-\pi/2, \pi/2]$ given by

$$\theta = \sin^{-1} x \text{ means exactly that } \sin\theta = x \text{ and } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

Proposition 8.4.2. *The derivative of $y = \sin^{-1} x$ is*

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}} \qquad (-1 < x < 1).$$

Proof. If $y = \sin^{-1} x$, then $\sin y = x$. Taking d/dx of both sides, we get

$$\cos y \frac{dy}{dx} = 1$$

and so

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

(We could also use the result on derivatives of inverse functions to get the same thing.) We want the answer in terms of x and we use

$$\cos^2 y + \sin^2 y = 1 \Rightarrow \cos^2 y = 1 - \sin^2 y \Rightarrow \cos y = \sqrt{1 - \sin^2 y}$$

(which is true because $\cos y \geq 0$ when $-\pi/2 \leq y \leq \pi/2$). So

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}.$$

We must exclude $x = \pm 1$ to avoid division by 0. $\square$

Note 8.4.3. Earlier, it was pointed out that $\sin^2 x$ means $(\sin x)^2$. Now we see that $\sin^{-1} x$ has nothing to do with a power of $\sin x$. Rather it means the 'inverse' function. One might argue that the notation is potentially misleading!

Exercise 8.4.4. What is the graph of $y = \sin^{-1} x$?

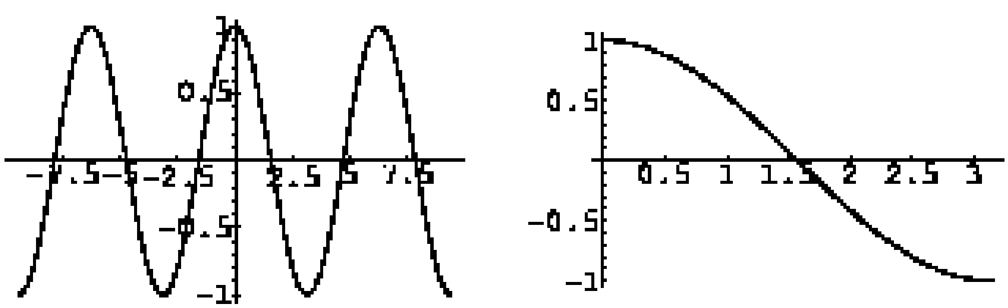
Definition 8.4.5. The '*inverse* $\cos$ *function*' (or arccos function) is the function $\cos^{-1} \colon [-1, 1] \to [0, \pi]$ given by

$$\theta = \cos^{-1} x \text{ means exactly that } \cos\theta = x \text{ and } 0 \leq \theta \leq \pi.$$

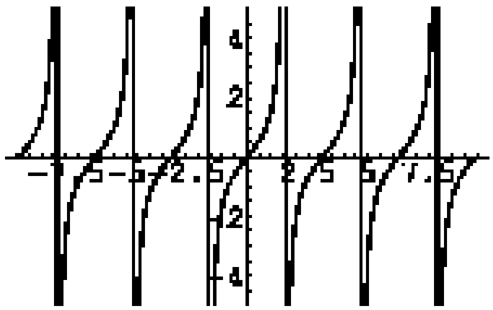
Proposition 8.4.6. *The derivative of* $y = \cos^{-1} x$ *is*

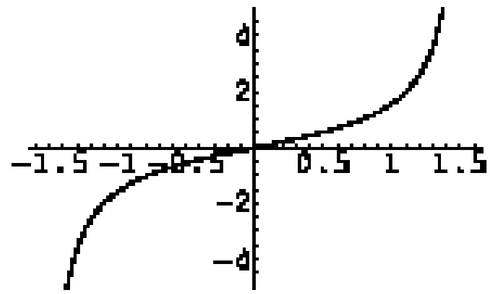
$$\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1 - x^2}} \qquad (-1 < x < 1).$$

The proof is very similar to the one for $(d/dx) \sin^{-1} x$. Here is the graph of $y = \cos x$ ($-3\pi \leq x \leq 3\pi$) and the restricted part of which we take the inverse function.



Here is the graph of $y = \tan x$ $(-3\pi \le x \le 3\pi)$ along with the section $-\pi/2 < x < \pi/2$. The apparently vertical parts of the graph of $y = \tan x$ should not be there at all. (That is because $\tan x$ is not defined for x an odd multiple of $\pi/2$.)





The section for $-\pi/2 < x < \pi/2$ is strictly monotone increasing and has an inverse function.

Definition 8.4.7. The '*inverse* tan *function*' (or arctan function) is the function $\tan^{-1} \colon \mathbb{R} \to (-\pi/2, \pi/2)$ given by

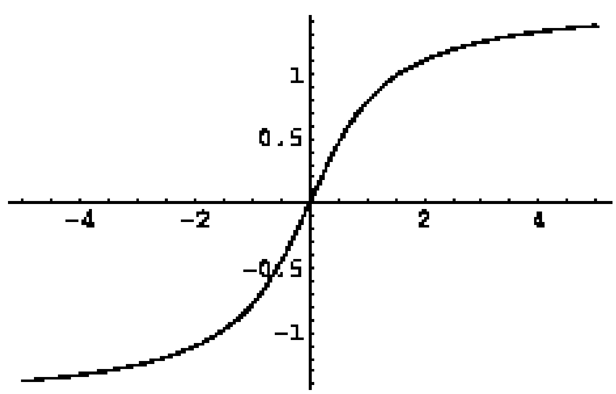
$$\theta = \tan^{-1} x \text{ means exactly that } \tan\theta = x \text{ and } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Proposition 8.4.8. *The derivative of* $y = \tan^{-1} x$ *is*

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} \qquad (x \in \mathbb{R}).$$

The proof is again not that different (but relies on $1 + \tan^2\theta = \sec^2\theta$).

Here is the graph $y = \tan^{-1} x$.



8.5 Hyperbolic functions

You might argue that this subsection does not fit here, and indeed it is a bit out of place. It relates to the exponential function.

Definition 8.5.1. The *hyperbolic cosine function* is denoted $\cosh x$ and is defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Definition 8.5.2. The *hyperbolic sine function* is denoted $\sinh x$ and is defined by

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

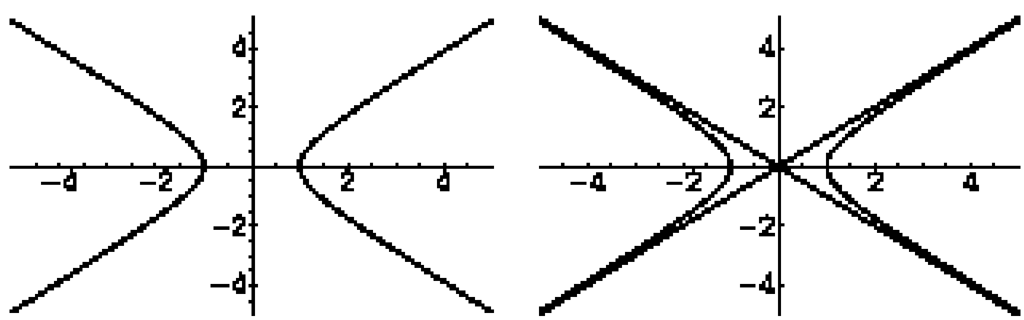
Proposition 8.5.3. *For every $x \in \mathbb{R}$,*

$$\cosh^2 x - \sinh^2 x = 1$$

Proof.

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{1}{4}(e^{2x} + e^{-2x} + 2e^x e^{-x} - (e^{2x} + e^{-2x} - 2e^x e^{-x})) \\
&= e^x e^{-x} = e^{x-x} = e^0 = 1 \quad \square
\end{aligned}$$

Remark 8.5.4. While $(\cos\theta, \sin\theta)$ lies on the unit circle, the above says that $(\cosh t, \sinh t)$ lies on the standard hyperbola $x^2 - y^2 = 1$. Here is the graph of that hyperbola, and a second version along with the two lines $y = \pm x$ (which are guiding lines that you can use to sketch the hyperbola). Since $\cosh t > 0$ always, $(\cosh t, \sinh t)$ lies on right half of the hyperbola. The value of t does not correspond to an angle in the picture.



Proposition 8.5.5.

$$\frac{d}{dx}\cosh x = \sinh x \quad \textit{and} \quad \frac{d}{dx}\sinh x = \cosh x.$$

Remark 8.5.6. A general principle is that for each formula about trig functions, there is a very similar formula for hyperbolic functions (but often with minus signs in different places). The hyperbolic functions are **not** periodic though.