

- It follows that  $f(x) > 0$  for all  $x$ .

*Proof.* This is not that easy. It really needs a result called the Intermediate Value theorem. That theorem concerns well-behaved functions on intervals (called continuous functions) that are positive somewhere and negative somewhere else and the conclusion is that the function must be zero in between. Since  $f(1) = 1 > 0$ , our function could never be negative.<sup>1</sup>  $\square$

- Since  $f'(x) = f(x) > 0$  for all  $x$ , it follows that  $f(x)$  is strictly monotone increasing on  $\mathbb{R} = (-\infty, \infty)$ .

In particular  $1 = f(0) < f(1)$ .

- For  $r \in \mathbb{Q}$ ,

$$\begin{aligned} \frac{d}{dx} f(rx)(f(x))^{-r} &= rf'(rx)(f(x))^{-r} + f(rx)(-r)(f(x))^{-r-1}f'(x) \\ &= rf(rx)(f(x))^{-r} - rf(rx)(f(x))^{-r-1}f(x) = 0 \end{aligned}$$

can be checked.

So

$$f(rx)(f(x))^{-r} = \text{constant}.$$

For  $x = 0$  we get

$$f(0)(f(0))^{-r} = \text{constant}$$

and since  $f(0) = 1$ , this tell us that the constant is 1. So

$$f(rx)(f(x))^{-r} = 1 \quad (\text{all } x \in \mathbb{R}).$$

Thus  $f(rx) = (f(x))^r$  (for  $r \in \mathbb{Q}$ ).

- In particular  $f(r) = f(1)^r$  for  $r \in \mathbb{Q}$ .

We define the number  $e$  to be  $e = f(1)$  and **define**  $e^x$  by

$$e^x = f(x).$$

Note that for  $x \in \mathbb{Q}$  (where we had a sensible way to define  $e^x$  previously) this is no change.

Another notation that is commonly used is  $\exp(x) = f(x)$ .

- With this definition, we have the law of exponents  $e^{x+a} = e^x e^a$  valid for all  $x, a \in \mathbb{R}$ .

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<sup>1</sup>Differentiable functions such as our  $f(x)$  are continuous.

*Proof.* Look at

$$g(x) = \frac{e^{x+a}}{e^x} = \frac{f(x+a)}{f(x)} = f(x+a)f(-x)$$

(with  $a$  fixed) and check that  $g'(x) = 0$ . So  $g(x)$  is constant. But  $g(0) = f(a)$  and so  $g(x) = f(a)$  for all  $x$ . Then we have  $f(x+a)/f(x) = f(a)$  or  $f(x+a) = f(x)f(a)$ , as we wanted.  $\square$

- We can compute

$$\frac{d}{dx} \frac{e^x}{x} = \frac{e^x x - e^x}{x^2} = \frac{e^x(x-1)}{x^2} > 0 \text{ for } x > 1$$

So  $e^x/x$  is (strictly) monotone increasing for  $x > 1$ . In particular, for  $x > 2$

$$\frac{e^x}{x} > \frac{e^2}{2}$$

and that means that with  $c = e^2/2$  we have

$$e^x > cx \quad (x > 2).$$

As  $c > 0$ , we can say that  $f(x) = \exp(x)$  becomes larger and larger with  $x$ .

Using a notation which we did not actually explain, it is in fact true that

$$\lim_{x \rightarrow \infty} e^x = \infty$$

(The meaning of this statement is this: if we fix any number  $M$  [which we think of as a ‘large’ number], then we can find  $x_0$  so that  $e^x > M$  holds whenever  $x > x_0$ . In less precise terms: we can ensure that  $e^x$  is close to  $\infty$  [or ‘large’] by making sure  $x$  is close enough to  $\infty$  [or just by making  $x$  larger than some  $x_0$ ].)

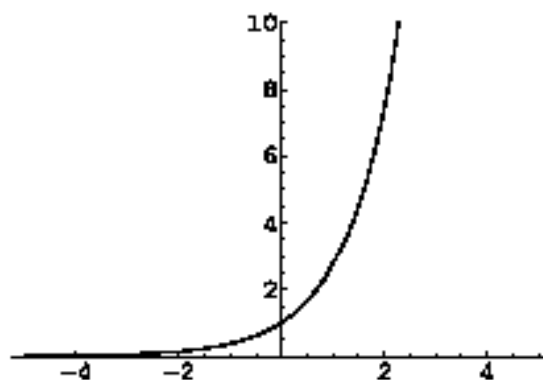
- Since  $e^{-x} = 1/e^x$ , we can then conclude that  $e^x$  becomes very small as  $x$  becomes large and negative. Using another notation we have not explained

$$\lim_{x \rightarrow -\infty} e^x = 0$$

(The meaning of this statement is this: if we fix any number  $\varepsilon > 0$  [which we think of as a ‘small’ positive number], then we can find  $x_0$  so that  $|e^x - 0| < \varepsilon$  holds whenever  $x < x_0$ .)

- Since we now know  $e^x > 0$  always,  $e^x$  is strictly monotone increasing and taking account of the two previous points, we can say that the graph of  $y = e^x$  must look something like this

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In[2]:= Plot[Exp[x], {x, -5, 5}, PlotRange -> {0, 10}]
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Out[2]= - Graphics -
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We have not come up with any way to estimate the value of  $e = f(1)$ , which influences the graph. The graph seems to run off the page quite quickly to the right and merge into the  $x$ -axis to the left. This is something we can check, at least roughly.

- For  $n \geq 1$  we can compute that

$$\frac{d}{dx} \frac{e^x}{x^n} = \frac{e^x x^n - e^x (n x^{n-1})}{x^{2n}} = \frac{e^x (x - n)}{x^{n+1}} > 0 \text{ for } x > n$$

So  $e^x/x^n$  is (strictly) monotone increasing for  $x > n$ . In particular, for  $x > n + 1$

$$\frac{e^x}{x^n} > \frac{e^{n+1}}{(n+1)^n}$$

and that means that with  $c_n = e^{n+1}/(n+1)^n$  we have

$$e^x > c_n x^n \quad (x > n + 1).$$

So, for  $n > 1$  and  $x > n + 2$  we have

$$\frac{e^x}{x^n} > \frac{c_{n+1} x^{n+1}}{x^n} = c_{n+1} x$$

and that implies

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^n} = \infty$$

(since  $c_{n+1} > 0$ ).

Conclusion: in the long run  $e^x$  grows faster than any power  $x^n$  as  $x \rightarrow \infty$ . Or ‘*exponential growth is fast!*’.

*Example 7.1* (Exponential growth/decay). There are several realistic situations where a differential equation

$$\frac{dy}{dx} = ky$$

arises. In some examples,  $x$  is time and so we might be better with

$$\frac{dy}{dt} = ky \tag{1}$$

where now  $y = y(t)$  depends on  $t$ . The examples include population growth (say of flies, or animals, where there is no food shortage or environmental change) where  $y(t)$  is the number in the population at time  $t$  and  $k$  is the difference between the birth rate and the death rate (per unit time). Another would be radioactive decay, where  $y(t)$  is the number of radioactive atoms in a sample at time  $t$  and  $|k|$  is the proportion that will decay (and then stop being radioactive) per unit time. Here  $k = -|k| < 0$ . In these examples, we need to assume the number is large so that considering  $y(t)$  to vary continuously (and not just through integer values) is not a big concern. Another example involves compound interest, where the compounding is done continuously (or over very small time steps).

To see what the solution of (1) are we rewrite it as

$$\frac{dy}{dt} - ky = 0$$

and then multiply across by  $e^{-kt}$  to get

$$e^{-kt} \frac{dy}{dt} - ke^{-kt} y = 0$$

The point of this is that now the left hand side is (by the product rule) the derivative of a product. So we get

$$\frac{d}{dt}(e^{-kt} y) = 0.$$

The factor  $e^{-kt}$  is known as an ‘*integrating factor*’ in this situation. We conclude

$$e^{-kt} y = c = \text{constant}$$

and so  $y = ce^{kt}$  gives all possible solutions.

Assuming  $y$  is a population, we need  $c > 0$ . If  $k > 0$  the graph of  $y$  against  $t$  looks more or less similar to the graph  $y = e^t$ . The factor  $k$  in the exponent re-scales the  $t$  axis by  $1/k$  while the  $c$  re-scales the  $y$ -axis.

If  $k < 0$ , then we need to reflect the exponential graph in the  $y$ -axis (and rescale the axes).

*Exercise 7.2.* Using an integrating factor trick, find all the solutions of the differential equation

$$\frac{dy}{dx} - 3y = e^{2x}.$$