### 4.3 Implicit differentiation

If we know an equation satisfied by y = f(x), like

$$2y^3 + xy^2 - x^2y + 1 = 0$$

we almost know what f(x) is. Not really, because for different x there could be several values of y.

However, suppose we do know we are talking about such a function (which then selects one solution y = f(x) for the equation for each x in the domain) and suppose also we somehow know that f'(x) = dy/dx exists. Then we can find dy/dx by differentiating the equation using the product rule and chain rule and so on as we go along. Remember y is not constant.

Differentiating the equation above gives

$$6y^{2}\frac{dy}{dx} + y^{2} + 2xy\frac{dy}{dx} - 2xy - x^{2}\frac{dy}{dx} = 0$$

Regrouping we get

$$(6y^2 + 2xy - x^2)\frac{dy}{dx} = -y^2 + 2xy$$

and so

$$\frac{dy}{dx} = \frac{-y^2 + 2xy}{6y^2 + 2xy - x^2}$$

So the value of dy/dx here depends on y as well as x. That is not usually what we want — usually we want dy/dx as a function of x — but in this case we are not sure which solution y = f(x) is, and so it makes sense that y comes in the result.

There is in fact a theorem that says you don't actually have to assume as much as the existence of dy/dx. If  $(x_0, y_0)$  is a solution of the equation, if you can differentiate the equation (implicitly) and if the formula for dy/dx does not involve division by zero, then there is in fact a differentiable y = f(x) that solves the equation on some domain including  $x_0$  and that also satisfies  $f(x_0) = y_0$ .

Here is an example. Take the equation  $x^2 + y^2 = 5$  and the point  $(x_0, y_0) = (2, -1)$ (which does satisfy the equation). Implicit differentiation gives (in this case)

$$\frac{dy}{dx} = -\frac{x}{y}$$

At (x, y) = (2, -1), that makes sense and is 2. The function we want is  $f(x) = -\sqrt{5 - x^2}$ and the domain where that is differentiable is  $-\sqrt{5} < x < \sqrt{5}$ . But note that the derivative of the square root goes beyond what we dealt with so far. It's now time to discuss that briefly.

#### 4.4 Fractional and rational powers

If  $q \in \mathbb{N}$ , then it makes sense to talk about the  $q^{\text{th}}$  root  $x^{1/q}$  of x for  $x \ge 0$ . For q odd we can talk about  $x^{1/q}$  for all  $x \in \mathbb{R}$ .

One way to see that is to rely on a graph of  $y = x^q$ .



For the case of even powers only horizontal lines  $y = y_0$  with  $y_0 \ge 0$  cross the graph. There are then (for  $y_0 > 0$ ) always two values of x with  $x^q = y_0$  (only x = 0 if  $y_0 = 0$ ) and we define

 $(y_0)^{1/q} = \text{ the } x \ge 0 \text{ with } x^q = y_0 \quad (q \text{ even}, y_0 \ge 0).$ 

For q odd, there is always just one  $x \in \mathbb{R}$  with  $x^q = y$  no matter what  $y \in \mathbb{R}$  we pick. We take

$$y^{1/q} = \text{ the } x \in \mathbb{R} \text{ with } x^q = y \quad (q \text{ odd}).$$

Now if we exchange the roles of y and x, we can say

$$x^{1/q} = y$$
 where  $y^q = x$  and if q is even we insist  $x \ge 0, y \ge 0$ 

That means that the graph of  $y = x^{1/q}$  is the same as the graph of  $y^q = x$  when q is odd, but only the part where  $y \ge 0$  if q is even. The graph of  $y^q = x$  can be got from the graph of  $y = x^q$  by reflecting in the line y = x. (See figure 4.1.)

While we are at it, we can say what  $x^r$  means for (nonzero) rational exponent  $r \in \mathbb{Q}$ . We should write r = p/q with  $p \in \mathbb{Z} \setminus \{0\}, q \in \mathbb{N}$  and have no common factors between p and q. Then, we define

 $x^r = x^{p/q} = (x^{1/q})^p$  where if q is even we insist  $x \ge 0$ .

We should not try to define  $0^0$ , but  $x^0 = 1$  when  $x \neq 0$  is safe. So is  $0^r = 0$  for r > 0.

With these definitions, the laws of exponents hold:

$$(x^{r})^{s} = x^{rs};$$
  $x^{r}x^{s} = x^{r+s};$   $x^{r}y^{r} = (xy)^{r}$ 

in the sense that if both sides makes sense, then they will have the same value. These laws work for x > 0, but you need to be careful when  $x \le 0$ . (For instance  $((-1)^2)^{1/4} = 1$  but  $(-1)^{1/2}$  is not good.)

Now to figure out the derivative of the  $q^{\text{th}}$  root function, or

$$\frac{d}{dx}x^{1/q},$$



Figure 4.1: graphs  $y = \sqrt{x}$ ;  $y = \sqrt{x}$  with  $y = x^2$  (the  $x \ge 0$  part only) and y = x;  $y = x^{1/3} = \sqrt[3]{x}$ ; and  $y = x^{1/3}$  with  $y = x^3$  and y = x.

we can use implicit differentiation. Let  $y = x^{1/q}$  and so  $y^q = x$ . **IF** we somehow know that dy/dx exists, we can use implicit differentiation to find

$$qy^{q-1}\frac{dy}{dx} = 1.$$

So we must have

$$\frac{dy}{dx} = \frac{1}{qy^{q-1}} = \frac{1}{q(x^{1/q})^{q-1}} = \frac{1}{q}\frac{1}{x^{1-1/q}} = \frac{1}{q}x^{(1/q)-1}$$

This ends up in terms of x, which is tidy. But there is a snag if x = 0. For one thing we are dividing by 0 in this case. And if q is even we are at the end point of the domain of the  $q^{\text{th}}$  root function (which is the interval  $[0, \infty)$ ). We have not considered what a derivative might mean at an end point and so we have another reason when q is even not to allow x = 0.

How could we know there is a derivative when  $x \neq 0$ ? Well, there is a theorem about inverse functions that covers this case, which is the correct way to prove it. Looking at the graphs, it seems reasonable that if a graph has a tangent line, then reflecting the graph in the diagonal line y = x should give another graph with a tangent line. That is true in a way, but for  $y = x^3$  the tangent line at the origin is y = 0 (slope 0). When you reflect that in the line y = x you get the vertical line x = 0, which has no slope. That explains in a graphical way why the cube root has no derivative at 0.

Finally, once we know the derivative of  $q^{\text{th}}$  root function, we can use the chain rule to figure out that

$$\frac{d}{dx}x^r = \frac{d}{dx}x^{p/q} = rx^{r-1}$$

for  $r = \frac{p}{q} \in \mathbb{Q}$  and x > 0. It also works for x < 0 if q is odd and even works for x = 0 if we have both q odd and r > 1.

### 5 Trigonometry

One big thing: Calculus only works right if we stick to radians.

We measure angles in radians, which means the arc length subtended on a unit circle centred at the vertex of the angle.



You can also take a circle of any radius and divide the arc length by the radius. That way you see that radians are dimensionless (as a ratio of lengths).

We extend the idea of angle to allow a direction of the angle and angles  $\theta > 2\pi$ . A simple geometrical way to say what  $\cos \theta$  and  $\sin \theta$  mean is this. Take a unit circle centred at the origin in the plane, which is the circle  $x^2 + y^2 = 1$ . Draw the angle  $\theta$  with vertex at the origin and starting from the positive x-direction, going anticlockwise if  $\theta > 0$ , clockwise if  $\theta < 0$ . Then the point on the unit circle where the angle ends has coordinates  $(x, y) = (\cos \theta, \sin \theta)$ . (See Figure 5.1.)



Figure 5.1: Definition of  $\cos \theta$  and  $\sin \theta$ :  $P = (\cos \theta, \sin \theta)$ 

From this approach, it is clear right away that  $\cos^2 \theta + \sin^2 \theta = 1$  (remember that  $\cos^2 \theta$  is a strange way of writing  $(\cos \theta)^2$ ). Another thing that is clear is the sign of  $\cos \theta$  and of  $\sin \theta$  in the different quadrants.

From the picture, it seems easy to believe

$$\lim_{\theta \to a} \cos \theta = \cos a \text{ and } \lim_{\theta \to a} \sin \theta = \sin a.$$

The key limit for calculus is one we mentioned in the first lecture

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

Using that and some trig identities about  $\cos(a+h)$  and  $\sin(a+h)$ , it is possible to prove quite easily that

$$\frac{d}{dx}\sin x = \cos x$$
 and  $\frac{d}{dx}\cos x = -\sin x$ .

Then the derivatives of the other trig functions  $\tan x = \frac{\sin x}{\cos x}$ ,  $\cot x$ ,  $\sec x$  and  $\csc x$  all follow:

$$\frac{d}{dx}\tan x = \sec^2 x, \quad \frac{d}{dx}\cot x = -\csc^2 x, \quad \frac{d}{dx}\sec x = \sec x\tan x, \quad \frac{d}{dx}\csc x = -\csc x\cot x$$

Just to emphasise about radians. If we worked in degrees we have this relation between degrees and radians

$$\theta^{\circ} = \frac{\pi}{180}\theta$$
 (radians).

 $\operatorname{So}$ 

$$\lim_{\theta \to 0} \frac{\sin(\theta^{\circ})}{\theta} = \lim_{\theta \to 0} \frac{\sin\left(\frac{\pi}{180}\theta\right)}{\theta} = \lim_{\theta \to 0} \frac{\pi}{180} \frac{\sin\left(\frac{\pi}{180}\theta\right)}{\frac{\pi}{180}\theta} = \frac{\pi}{180}$$

That says that  $f(x) = \sin(x^\circ)$  has  $f'(0) = \pi/180$  (not the same at all as  $\cos 0$ ). In general  $f(x) = \sin(x^\circ)$  has  $f'(x) = (\pi/180)\cos(x^\circ)$  and so calculus with trigonometric functions in degrees would be very different.

# 6 Sign of dy/dx

This is perhaps a first application of derivatives. At least it has practical applications to finding largest and smallest values.

### 6.1 Monotone functions

**Definition 6.1.1.** Let  $f: A \to \mathbb{R}$  be a function defined on a domain  $A \subset \mathbb{R}$ . Then f(x) is called *monotone increasing* (on A) if whenever  $x_1 < x_2$  (with  $x_1, x_2 \in A$ ) then  $f(x_1) \leq f(x_2)$ .

A way to think about this is to say that if you move left to right on the graph y = f(x), you always end up at a higher point (or at least at a point no lower) than where you started.

(You might justifiably think that non-decreasing is a more descriptive term.)

**Definition 6.1.2.** Again let  $f: A \to \mathbb{R}$  be a function defined on a domain  $A \subset \mathbb{R}$ . Then f(x) is called *strictly monotone increasing* (on A) if whenever  $x_1 < x_2$  (with  $x_1, x_2 \in A$ ) then  $f(x_1) < f(x_2)$ .

(This means that we definitely get to a higher point if we move left to right.)

Remark 6.1.3. We define monotone decreasing by saying  $f(x_1) \ge f(x_2)$  when  $x_1 < x_2$  (and  $f(x_1) > f(x_2)$  for strictly monotone decreasing).

Sometimes it helps to know that if f(x) is monotone decreasing on some set, then g(x) = -f(x) is monotone increasing.

A function is called monotone (on a set A contained in its domain) if it is *either* monotone increasing *or* monotone decreasing.

**Proposition 6.1.4.** If y = f(x) is defined and differentiable on an open interval, if it is also monotone increasing, then  $f'(x) \ge 0$  for all x in the interval.

The proof of this is quite straightforward just using the definition of f'(x) and the definition of monotone increasing.

*Examples* 6.1.5. (i) The function  $f \colon \mathbb{R} \to \mathbb{R}$  given by  $f(x) = x^3$  is strictly monotone increasing. (If  $x_1 < x_2$  then we always have  $x_1^3 < x_2^3$ . It may help to look at the graph of  $y = x^3$  again.)

Now  $f'(x) = 3x^2$  and although  $f'(x) \ge 0$  is true, we can't say f'(x) > 0 always since f'(0) = 0.

So a 'strictly monotone increasing' version of the proposition (with derivative > 0) is **not** true.

(ii) Consider the function f(x) = -1/x, a function which naturally makes sense for all  $x \neq 0$ . So we should take  $A = \{x \in \mathbb{R} : x \neq 0\}$  (every x except x = 0) and define  $f: A \to \mathbb{R}$  by the rule f(x) = -1/x.

Then  $f'(x) = 1/x^2$  and we see that it is always positive. However, if we look at the graph, we can see that the function is *not increasing* overall. Look at  $x_1 = -1$ ,  $x_2 = 1$ , where  $x_1 < x_2$  but  $f(x_1) = 1 > f(x_2) = -1$ .

**Theorem 6.1.6.** If y = f(x) is differentiable (f'(x) exists) for all x in an interval and if f'(x) > 0 for all x in the interval, then f(x) is **strictly monotone increasing** on that interval.

If instead we assume  $f'(x) \ge 0$  on the interval, then f(x) is monotone increasing on the interval. Although this may seem reasonably obvious, the previous example shows that it is not so obvious. We need the assumption about intervals for it to be true.

It can be *proved* to be correct using a theorem called the Mean Value Theorem (for derivatives).

**Corollary 6.1.7.** Assume f is a differentiable function. Then the intervals where  $f'(x) \ge 0$  are exactly the intervals where f(x) is monotone increasing.

Applying the foregoing to -f(x) in place of f(x) we find:

**Corollary 6.1.8.** The intervals where  $f'(x) \leq 0$  are the intervals where f(x) is monotone decreasing.

- *Examples* 6.1.9. (a) Find the intervals where  $f(x) = x^3 2x^2 + x + \frac{1}{2}$  is increasing and the intervals where it is decreasing.
- (b) Find the intervals where  $f(x) = \frac{5x-8}{(x-2)^2}$  is increasing and the intervals where it is decreasing.

**Corollary 6.1.10.** If f(x) satisfies f'(x) = 0 for each x in an interval, then f(x) is constant on the interval.

*Proof.* From f'(x) = 0 we get  $f'(x) \ge 0$  always. So f(x) is monotone increasing on the interval.

But we also get  $f'(x) \leq 0$  from f'(x) = 0 and that tells us that f(x) is also monotone decreasing on the interval.

The only functions that are both monotone increasing and monotone decreasing are constants.  $\hfill \Box$ 

### 6.2 Max and min points

**Definition 6.2.1.** A point x = c is an absolute maximum point for the function y = f(x) if  $f(c) \ge f(x)$  for every x in the domain of f.

A point x = c is an absolute minimum point for the function y = f(x) if  $f(c) \le f(x)$  for every x in the domain of f.

Remark 6.2.2. It is quite easy to understand the importance of finding these points. If we imagine that f(x) is the cost of building something and x is a parameter that can be controlled, then it would be natural to want to minimise the cost by choosing the most efficient x. For other problems, you might want to choose x so as to get the most out if f(x) represents output.

You can also imagine problems where there is more than one thing to vary. That would amount to a function of more than one variable and we will get around to dealing with such functions. **Definition 6.2.3.** Let y = f(x) be a function. Then a point x = c is a *local max point* (or *relative max point*) for f(x) if  $f(c) \ge f(x)$  for all x nearby c

[Technically, we should say  $f(c) \ge f(x)$  for all x within some positive distance of c on either side of c]

A point x = c is called a *local min point* (or a *relative min point*) for f(x) if  $f(c) \le f(x)$  for all x nearby c.

*Remark* 6.2.4. Although absolute max or min values are probably what you would mostly like to find in applications, local max or min points are what you can easily find with calculus.

Usually there are not so many local max points and if there is any absolute max point, it should be one of the local max points. In this way calculus helps a lot.

The next results explain how it helps.

**Definition 6.2.5.** A critical point for a function y = f(x) is a values of x where f'(x) = 0.

**Theorem 6.2.6.** Suppose y = f(x) has a local max (or min) at x = c. If f is differentiable at x = c and x = c is an interior point of the domain of f (that is not an end point) then f'(c) = 0.

## 7 Exponential function

One way to define the function  $f(x) = \exp(x) = e^x$  is to say that it is the only function  $f: \mathbb{R} \to \mathbb{R}$  with the properties

$$f'(x) = f(x)$$
 and  $f(0) = 1$ .

It is perhaps not a good way to define it but it is a valid way.

What it means is that the function  $y = e^x$  solves a differential equation

$$\frac{dy}{dx} = y$$

The drawback of the approach here is how to know that there is a solution of the differential equation (apart from f(x) = 0 which won't satisfy f(0) = 1). Leaving this aside, it is not hard to check that

• 
$$\frac{d}{dx}(f(x)f(-x)) = f'(x)f(-x) + f(x)f'(-x)(-1) = f(x)f(-x) - f(x)f(-x) = 0.$$

- So f(x)f(-x) is constant. Looking at x = 0 we find that the constant is f(0)f(0) = 1.
- From f(x)f(-x) = 1 for all x, we can conclude that f(x) is never 0.

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