2.2 Limits of polynomials and rational functions

Polynomial functions include examples such as

$$f(x) = 17x^2 + 5x - 198$$

or

$$f(x) = 123x^4 + 6x^3 - x^2 + 16x - 11.$$

In general a polynomial is a finite sum of constants times powers of the variable.

So the general form is

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are all numbers (real numbers in our case).

Functions give by a polynomial formula are well-behaved, at least in the sense that they make sense for every $x \in \mathbb{R}$.

Polynomials are nice from the point of view of limits. If p(x) is a polynomial and $a \in \mathbb{R}$ is any point, then

$$\lim_{x \to a} p(x) = p(a)$$

(This can be proved from the Theorem above, using the two example limits.)

Rational functions are functions given by a formula

$$f(x) = \frac{p(x)}{q(x)}$$

where p(x) and q(x) are both polynomials and q(x) is not identically zero. So they are like fractions of polynomials.

Rational functions are quite nice also. They may not make sense for all $x \in \mathbb{R}$ — we have to avoid any x that makes the denominator q(x) = 0.

Limits of rational functions are not complicated (as long as we stay away from zeros of the denominator).

If $f(x) = \frac{p(x)}{q(x)}$ is a rational function and $a \in \mathbb{R}$ is a point where $q(a) \neq 0$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{p(x)}{q(x)} = \frac{\lim_{x \to a} p(x)}{\lim_{x \to a} q(x)} = \frac{p(a)}{q(a)} = f(a)$$

Sometimes rational functions can have limits at points where the denominator is 0. If q(a) = 0, but $p(a) \neq 0$,

$$\lim_{x \to a} \frac{p(x)}{q(x)}$$

will not exist (at least not in the ordinary sense of finite limits). Reason: For x near a (but $x \neq a$) q(x) will be close to q(a) = 0, but p(x) will be close to p(a). So p(x)/q(x) will be very large in absolute value.

On the other hand, if p(a) = q(a) = 0, there is a chance that $\lim_{x\to a} \frac{p(x)}{q(x)}$ exists. In fact, there is a theorem called the **Remainder Theorem** that says: MA1131 - Lecture 2 (1/10/2010)

If p(x) is a polynomial and p(a) = 0, then x - a divides p(x).

Maybe an even clearer way to say it: If p(x) is a polynomial and p(a) = 0, then there is another polynomial P(x) with

$$p(x) = (x - a)P(x)$$

for each x.

So if p(a) = q(a) = 0, we can factor out x - a from each of p(x) and q(x) and rewrite

$$\frac{p(x)}{q(x)} = \frac{(x-a)P(x)}{(x-a)Q(x)} = \frac{P(x)}{Q(x)}$$

The fraction P(x)/Q(x) is 'simpler' than p(x)/q(x) and there is a chance that $Q(a) \neq 0$.

Since
$$f(x) = \frac{p(x)}{q(x)}$$
 can be equally well written $f(x) = \frac{P(x)}{Q(x)}$ for $x \neq a$, we have

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \lim_{x \to a} \frac{P(x)}{Q(x)}$$

If P(a) = Q(a) = 0 we can divide out by x - a again, until we reach a fraction where the numerator and denominator are not both 0 at x = a. We've discussed that situation just above.

Exercises: Find $\lim_{x \to \sqrt{2}} \frac{2x^2 - 3x + 6}{x^2 + 2}$ and $\lim_{x \to 2} \frac{x^2 - 6x + 8}{x^2 - 4}$.

3 Derivatives (in one variable)

3.1 The definition

For calculus (differentiation anyhow) it is easier to stick to functions f(x) defined on open intervals (α, β) . For any $a \in (\alpha, \beta)$ we can consider

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

because it at least satisfies the first requirement: the quantity

$$\frac{f(x) - f(a)}{x - a}$$

makes sense for x near a on either side of a. So, there is at least a chance that the limit exists.

Definition 3.1.1. If f(x) is defined on an open interval that includes a, then we say that the function f(x) is differentiable at a if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists (as a finite quantity). If the limit exists, we denote it by f'(a).

An equivalent way to define it is this:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(if the limit exists (and is finite)).

3.2 Interpretations/motivation

Notice that $\frac{f(x) - f(a)}{x - a}$ is the slope of the line joining the point (a, f(a)) on the graph of f to the point (x, f(x)) — at least it is that as long as $x \neq a$.



In this picture a = 1 and x = 1.5 (or it could be the other way around a = 1.5 and x = 1).

From this point of view, as x approaches a (on either side of a, but we don't allow x = a), the chord seems to become closer and closer to being a tangent to the graph at the point on the graph where x = a.

In fact, we turn the picture around and use it as a definition of a tangent line. We say that

If f(x) is a function defined on an open interval that includes a, and if f'(a) exists, then the *tangent line to the graph* y = f(x) at the point where x = a is the line through the point (a, f(a)) with slope f'(a).

We can work out the equation and conclude that the tangent line to the graph y = f(x)at the point where x = a is the line

$$y = f(a) + f'(a)(x - a)$$

This formula forms the basis of the linear approximation formula.

3.3 The linear approximation formula

 $f(x) \cong f(a) + f'(a)(x-a) \text{ for } x \text{ near } a.$ The line y = f(a) + f'(a)(x-a) (which is the tangent line to the graph y = f(x) at the point (a, f(a)) is in a certain sense the best linear approximation to the graph y = f(x)near x = a. We have to explain in what sense it is the best.

Notice that it is exactly true when x = a, as f(x) = f(a) + f'(a)(x - a) for x = a. But any line of any slope that goes through the point (a, f(a)) would have that property.

We can rearrange the definition of f'(a) as follows. First the usual definition

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$
$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} - f'(a) = 0$$
$$\lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x - a)}{x - a} = 0$$
$$\lim_{x \to a} \frac{f(x) - \left(f(a) + f'(a)(x - a)\right)}{x - a} = 0$$

What this tells us is that the discrepancy between the y-value y = f(x) of the point (x, y) = (x, f(x)) on the actual graph and the y-coordinate of the point (x, f(a) + f'(a)(x - a))a)) on the tangent line is

small even compared to the small quantity x - a

when x is near enough to a.

No other line has this property. to say that there is a derivative actually means exactly that there is a slope m so that the line y = f(a) + m(x - a) satisfies

$$\lim_{x \to a} \frac{f(x) - \left(f(a) + m(x-a)\right)}{x-a} = 0$$

The line y = f(a) + m(x - a) is the line of slope m through the point (a, f(a)).

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