MA1131: Advanced calculus

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Brief outline

Techniques, rather than theory.

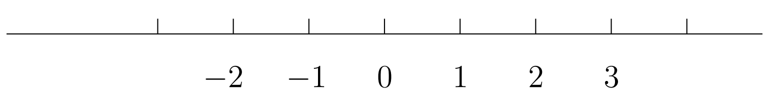
Topics: One variable (differentiation & integration), simple ODEs, partial derivatives, multiple integrals.

Continuous assessment. Tutorials. (Online) homework. Class tests Fridays **November 5** and **December 17**.

1 Basic notation and terminology

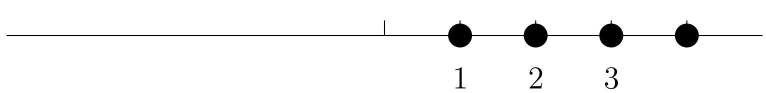
1.1 Real numbers $\mathbb{R}$

Positive, negative and 0 included. Picture as points on an axis.



1.2 Subsets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ of $\mathbb{R}$

Natural numbers $1, 2, 3, \ldots$. Set denoted $\mathbb{N} = \{1, 2, 3, \ldots\}$ (some people would include 0)



Integers $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$ (whole numbers)
Rationals $\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$ (fractions)

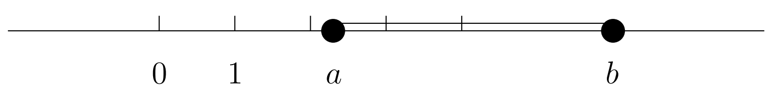
There are real numbers that are not in $\mathbb{Q}$ — such as $\sqrt{2}$, π and e. Called *irrational*. All numbers in $\mathbb{R}$ can be represented by decimals and rationals are those with repeating decimals.

1.3 Intervals

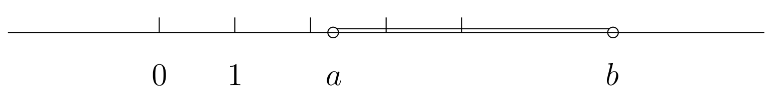
Subsets of the real numbers with no 'gaps' are called intervals.

There are several kinds.

(i) Finite closed intervals $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ (where $a \leq b$)



(ii) finite open intervals $(a, b) == \{x \in \mathbb{R} : a < x < b\}$ (where $a < b$)



(iii) half open (or half closed) intervals $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$

(iv) infinite intervals $(-\infty, b)$, $(-\infty, b]$, $(-\infty, \infty) = \mathbb{R}$, (a, ∞) and $[a, \infty)$

1.4 Functions

Most of the time we will deal with functions given by a formula — like

$$f(x) = \frac{x+1}{x-1} \text{ or } f(x) = xe^x \sin x$$

Formally a function $f\colon A \to B$ is a rule that associates *one and only one* element $f(a) \in B$ to *each* $a \in A$. Here A and B can be any sets.

Even more formal definitions are possible. For us, we will have A and B be sets of numbers (subsets of $\mathbb{R}$) to begin with. Later we will have slightly more complicated examples.

Strictly, to say what function you mean, you should say what the set A is (called the *domain* of the function), what the set B is, and what the rule is corresponding to our function (which we are calling f so far). In this approach, when we just write $f(x) = \frac{x+1}{x-1}$ we are being sloppy. However, from the rule that says what $f(x)$ is, we can probably assume that x is a real number and that we better not allow x to be 1 — because then the rule would not make sense.

So we might assume that, if the rule is $f(x) = \frac{x+1}{x-1}$, then the domain is probably the largest one where the rule makes sense — which is $A = \{x \in \mathbb{R} : x \neq 1\} = (-\infty, 1) \cup (1, \infty)$ in this case.

And we typically take $B = \mathbb{R}$ as our functions will have numerical values.

Two very simple examples: (but with little details to watch for)

(i) The square root function $\sqrt{x}$ means the positive square root. So $\sqrt{4} = 2$ (not -2).

The domain is $[0, \infty)$ as we don't want complex numbers at the moment.

(ii) The absolute value function

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

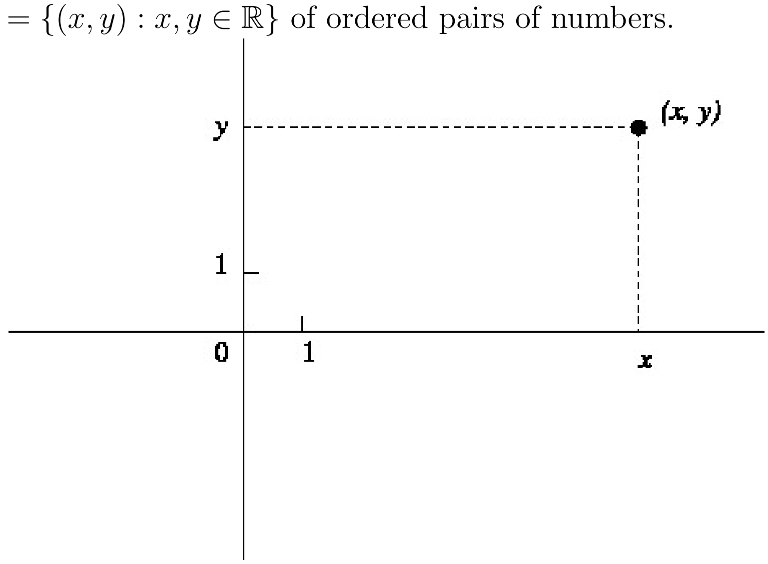
If you think about it, you will see it works when explained like this. $|-2|$ is 2 and that is the same as $-(-2)$.

We could also say $|x| = \sqrt{x^2}$. (Think that through!)

1.5 Functions have graphs

At least they do if the domain and range are sets of numbers.

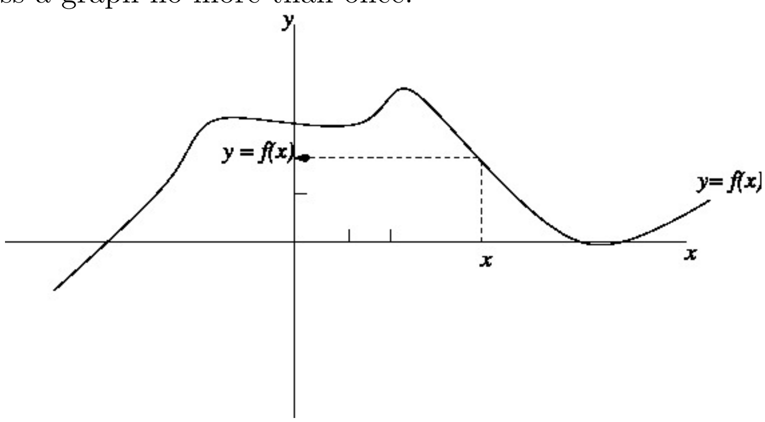
We picture the plane (with two perpendicular axes fixed in it for reference) as the set $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ of ordered pairs of numbers.



If $f\colon A \to B$ is a function where $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$), the graph of f is

$$\{(x, y) \in \mathbb{R}^2 : y = f(x)\}$$

(So the x-coordinates of all points on the graph are in A, the domain of f. If we know the graph, we know the domain too — or we can read it off the graph.) A vertical line can cross a graph no more than once.



Exercise: Graph $y = x^2$, $y = \sqrt{x}$ and $y = |x|$.

2 Limits

Formal treatment in analysis (MA1121). *Rather informal version*: For $a \in \mathbb{R}$ and $\ell \in \mathbb{R}$

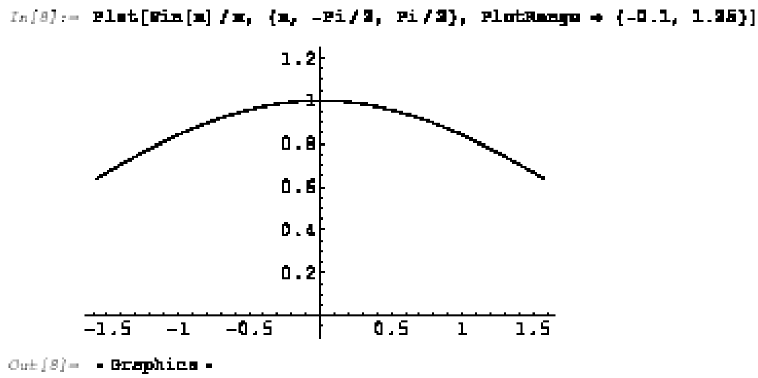
$$\lim_{x \to a} f(x) = \ell$$

means:

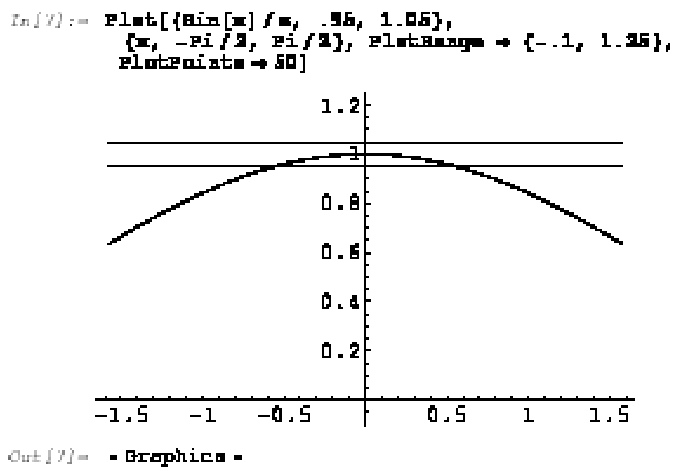
1. the domain of the function $f(x)$ includes all x within some positive distance from a, except not necessarily a itself
2. $f(x)$ will be close to ℓ if x is close enough to (but not equal to) a.
   A little more precisely: can ensure that $f(x)$ is within any specified positive distance from ℓ just by taking x within some positive distance of a (but $x = a$ is not considered).

2.1 Examples of limits

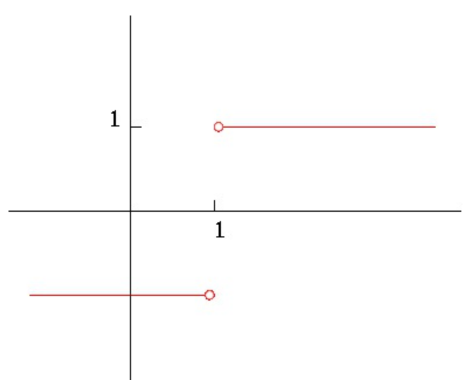
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$



$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$



$$\lim_{x\to 1} \frac{|x-1|}{x-1}$$



(In this case there is no limit.)

2.2 Basic results on limits

Lemma 2.2.1. *1. For any $a \in \mathbb{R}$, and any (constant) $\lambda \in \mathbb{R}$, $\lim_{x\to a} \lambda = \lambda$.*

2. *For any* $a \in \mathbb{R}$, $\lim_{x\to a} x = a$.

(These two are easy to verify directly from the definition. They are also clear in an informal way. Formal proof in MA1121.)

Theorem 2.2.2. *Suppose f and g are are $\mathbb{R}$-valued functions defined on subsets of $\mathbb{R}$. Suppose also $a, \ell, m \in \mathbb{R}$, $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = m$. Then*

(i) $\lim_{x\to a}(f(x) + g(x)) = \ell + m$

(ii) $\lim_{x\to a} f(x)g(x) = \ell m$

(iii) if $m \neq 0$, $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\ell}{m}$