UNIVERSITY OF DUBLIN

MA4151

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

JS Mathematics **Trinity Term 2001**
SS Mathematics

COURSE: 415 — FOURIER ANALYSIS AND WAVELETS

SAMPLE PAPER

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Answer 6 questions. Please use separate answer books for sections A and B.
Logarithmic tables are available if needed.

Section A

1.

2.

3.

4.

5. For sample questions on section A, see Professor Aron's exercise sheets.

Section B

6. (a) Define operators T_a, D_λ ($a \in \mathbb{R}, \lambda > 0$) on $L^2(\mathbb{R})$ by $(T_a f)(x) = f(x-a)$ and $(D_\lambda f)(x) = \sqrt{\lambda} f(\lambda x)$. Verify that each of the operators is an isometry of $L^2(\mathbb{R})$ and that they satisfy $T_a D_\lambda = T_{a/\lambda} D_\lambda$.
For $\mathcal{F}$ the Fourier transform on $L^2(\mathbb{R})$, show that $\mathcal{F} D_\lambda = D_{1/\lambda}\mathcal{F}$.

   (b) Assuming that $\phi \in L^2(\mathbb{R})$ satisfies $\phi = \sum_{k=k_0}^{k_1} c_k D_2 T_k \phi$ and that $\{T_k\phi : k \in \mathbb{Z}\}$ are orthonormal in $L^2(\mathbb{R})$, show that

$$\sum_{k\in\mathbb{Z}} c_k \overline{c_{k-2\ell}} = \begin{cases} 1 & \text{for } \ell = 0 \\ 0 & \text{for } \ell \in \mathbb{Z}, \ell \neq 0. \end{cases}$$

   (c) Show that $\phi(x) = x\chi_{[0,1)}(x) + (1-x)\chi_{[1,2)}$ satisfies

$$\phi = \frac{1}{2\sqrt{2}} D_2 T_0 \phi + \frac{1}{\sqrt{2}} D_2 T_1 \phi + \frac{1}{2\sqrt{2}} D_2 T_2 \phi$$

7. (a) What is meant by a *compactly supported* $L^2(\mathbb{R})$ function?
Show that a compactly supported $L^2(\mathbb{R})$ function must be in $L^1(\mathbb{R})$.

(b) If a dilation equation $\phi = \sum_{k=k_0}^{k_1} c_k D_2 T_k \phi$ has a compactly supported continuous solution $\phi \in L^2(\mathbb{R})$, show that it has its support in $[k_0, k_1]$.

8. (a) Explain the method of constructing wavelet bases of $L^2(\mathbb{R})$ based on a *multiresolution analysis*.

(b) If ϕ satisfies a two scale dilation equation

$$\phi = \sum_k c_k D_2 T_k \phi,$$

has orthonormal translates $T_k\phi$ ($k \in \mathbb{Z}$), gives rise to a multiresolution analysis by V_0 = the closure of the span of $\{T_k\phi : k \in \mathbb{Z}\}$, $V_n = D_{2^n} V_0$ ($n \in \mathbb{Z}$) and if $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then show that

$$\left| \int_{-\infty}^{\infty} \phi(x)\, dx \right| = 1$$

9. (a) Define convergence for an infinite product

$$\prod_{n=1}^{\infty} a_n.$$

Also define absolute convergence and explain how to prove that absolute convegence implies convergence.

(b) If $p(\xi) = \sum_{k=k_0}^{k_1} c_k e^{-2\pi i k \xi}$ is a trigonometric polynomial with $p(0) = \sqrt{2}$ and $|p(\xi)|^2 + |p(\xi + 1/2)|^2 \equiv 2$, then show that

$$\hat{\phi}(\xi) = \prod_{n=1}^{\infty} \frac{p(\xi/2^n)}{\sqrt{2}}$$

defines a function $\hat{\phi} \in L^2(\mathbb{R})$

Course 415 Exercise Set IV
R. M. Aron 8 January, 2001

18. Let X and Y be two arbitrary sets. Prove that there is an onto function $f : X \to Y$ or that there is an onto function $f : Y \to X$.

19. Calculate $\min_{a,b\in\mathbb{R}} \int_0^1 (x^2 - a - bx)^2 dx$.

20. Calculate $\max \int_0^1 x^2 g(x)dx$, where the maximum is taken over all measurable functions $g : [0, 1] \to \mathbb{R}$ such that $\int_0^1 g(x)dx = \int_0^1 xg(x)dx = 0$ and $\int_0^1 g^2(x)dx = 1$. Is there a relation between this question and problem 19?

21. Let $f \in L_1(\mathbb{R})$. Show that the function $t \in \mathbb{R} \rightsquigarrow \hat{f}(t) \in \mathbb{C}$ is continuous. (Hint: Let $t_n \to t_0 \in \mathbb{R}$. Use the definition of the Fourier transform and the Lebesgue dominated convergence theorem to show that $\hat{f}(t_n) \to \hat{f}(t_0)$.)

22. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function whose support is contained in $[A, B]$ (i.e. $f(t) = 0$ for $t \notin [A, B]$). For $s \in \mathbb{R}$, define $f_s : \mathbb{R} \to \mathbb{R}$ by $f_s(t) \equiv f(s + t)$. Prove that $s \in \mathbb{R} \rightsquigarrow f_s \in L_1(\mathbb{R})$ is continuous.

23. For an arbitrary real number $\delta > 0$, let $g_\delta : \mathbb{R} \to \mathbb{R}$ be a continuous function with the following properties: $g_\delta(t) \geq 0$ for all $t \in \mathbb{R}$, $g_\delta(t) = 0$ for all $t \in \mathbb{R}$ such that $|t| \geq \delta$, and $\int_{-\infty}^{\infty} g_\delta(t)dt = 1$.

(a). Sketch what the graph of such a function g_δ should look like.

(b). Prove that for any $f \in C(\mathbb{R})$ and any $t_0 \in \mathbb{R}$, $f * g_\delta(t_0) \to f(t_0)$ as $\delta \to 0$.

24. Let f and g be in $L_1(\mathbb{R})$. Prove that $f * g = g * f$.

25. Fix $k \in \mathbb{N}$. Calculate $\hat{f}_k$ for $f_k : \mathbb{R} \to \mathbb{R}$ given by

$$f_k(t) = \begin{cases} 1 & \text{if } |t| \leq k \\ 0 & \text{if } |t| > k \end{cases} .$$

Find $f_1 * f_2$ and $\widehat{f_1 * f_2}$.

Course 415 Exercise Set III
R. M. Aron 15 November, 2000

12. **Theorem.** All subsets of $\mathbb{R}$ are countable.
Proof. Let $\mathcal{S} \equiv \{S \subset \mathbb{R} \ : \ S \text{ is countable } \}$.
1. *Clearly* $\mathcal{S}$ is partially ordered by the relation $S_1 \preceq S_2 \iff S_1 \subseteq S_2$ (for $S_1, S_2 \in \mathcal{S}$).
2. *Clearly* $\mathcal{S}$ is inductive with this partial ordering. That is, if $\mathcal{C} \subset \mathcal{S}$ is a chain of elements in $\mathcal{S}$, then this chain has an upper bound.
3. By Zorn's Lemma, there is a maximal element M of $\mathcal{S}$.
4. *Clearly* $M = \mathbb{R}$. Indeed, if $M \subsetneq \mathbb{R}$, then there is a point $t_0 \in \mathbb{R}\backslash M$. However, since M is countable so is $M_1 = M \cup \{t_0\}$. And, *clearly* $M_1 \in \mathcal{S}$ is a strictly larger element than the maximal element M.
This contradiction completes the proof. Of course, the preceding result is false. However, can you (i) prove each of the *clear* steps above which is true and (ii) disprove each of these steps which is false.

13. Show that $\ell_p \subset \ell_q$ for $1 \leq p \leq q \leq \infty$. Is the same set inclusion true for L_p and L_q, where we take $X = [0,1]$ and Lebesgue measure? What if we take $X = \mathbb{R}$?

14.(a). Let $A \subset L_1[0,1]$ be defined as follows:

$$A = \{f \in L_1[0,1] \ : \ \int_{[0,1]} f(t)dm(t) = 1\}.$$

Prove that A is closed and convex in $L_1[0,1]$. Show that A has infinitely many elements of minimal norm.
(b). Let $A \subset C[0,1]$ be defined as follows:

$$A = \{f \in C[0,1] \ : \ \int_{[0,\frac{1}{2}]} f(t)dm(t) - \int_{[\frac{1}{2},1]} f(t)dm(t) = 1\}.$$

Prove that A is closed and convex in $C[0,1]$, but that A has no elements of minimal norm.

15. Let $S \subset H$ be an arbitrary subset of Hilbert space H. Define

$$S^\perp \equiv \{x \in H \ : \ < x, s >= 0 \text{ for all } s \in S\}.$$

(a). Let $H = \mathbb{R}^2$, and let $S = \{(1,1),(1,3)\}$. Calculate $S^\perp$. Repeat for $H = \ell_2$ and $S = \{e_{2n} \ : \ n \in \mathbb{N}\}$.
(b). Prove that $S^\perp$ is a closed subspace of H.
(c). Show that $S \subset S^{\perp\perp}$. Under what conditions do we have equality?

16. Show that c (of exercise 7) and c_0 are isomorphic. That is, show that there is a linear, continuous, bijective mapping $T : c \to c_0$.

17. Suppose that H is a Hilbert space and that (e_j) is an orthonormal sequence in H (i.e. for each j and k, $< e_j, e_k >= \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$). Prove that (e_j) is a basis for H if and only if the vector space span of $\{e_j \ : \ j \in \mathbb{N}\}$ is dense in H.

Course 415 Exercise Set II
R. M. Aron 13 October, 2000

6.(a). Calculate $\lim_{m\to\infty}\lim_{n\to\infty}|\cos(m!\pi x)|^n$.
(b). Why did I ask this question?

7. Let $c \equiv \{x = (x_j) \ : \ \lim_{j\to\infty} x_j$ exists $\}$. Show that c is a vector space, and that it is a Banach space with the norm $||x|| = \sup_j |x_j|$. Find a basis for c.

8. Consider the following mapping $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$.

$$T(x_1, x_2, x_3, ...) \equiv (\frac{x_1 + x_2}{\sqrt{2}}, \frac{x_3 + x_4}{\sqrt{2}}, ..., \frac{x_{2n-1} + x_{2n}}{\sqrt{2}}, \frac{x_1 - x_2}{\sqrt{2}}, ..., \frac{x_{2n-1} - x_{2n}}{\sqrt{2}}).$$

Prove that T is linear and also that T is an *isometry*, that is that $||T(x)|| = ||x||$ for every $x = (x_1, .., x_{2n}) \in \mathbb{R}^{2n}$.

9. Let X be a Banach space with basis $\{e_j\}$. Prove that the mapping $p_i : X \to \mathbb{K},\ p_i(x) = x_i$ is continuous. (Here, $x = \sum_{j=1}^{\infty} x_j e_j$.)

10. Recall that the space of continuous real-valued functions on $[0, 1]$, $C[0, 1]$, is complete (i.e. a Banach space) when it is endowed with the norm $||f|| = \max_{0\le t\le 1} |f(t)|$. Let $P[0, 1] \subset C[0, 1]$ be the subspace consisting of the restrictions to $[0, 1]$ of all polynomials.
(a). Find a *discontinuous* linear form $\phi : P[0, 1] \to \mathbb{R}$.
(b*). Find a *discontinuous* linear form $\phi : C[0, 1] \to \mathbb{R}$.

11. Let $(f_j)_{j=1}^{\infty}$ and f be functions on $[0, 1]$. Consider the following properties:
(a). $f_j \to f$ in $L_\infty[0, 1]$.
(b). $f_j \to f$ in $L_2[0, 1]$.
(c). $f_j \to f$ in $L_1[0, 1]$.
(d). $f_j(t) \to f(t)$ for every $t \in [0, 1]$.
(e). $f_j \to f$ almost everywhere.

Find all possible relations between these 5 different types of convergence. Where an implication is true, prove it. Where it is false, provide a counterexample.

Course 415 Exercise Set I
R. M. Aron 13 October, 2000

1. A set S in a metric space (X, d) is called a G_δ provided S can be written as a countable intersection of open sets; that is, $S = \cap_{n=1}^{\infty} U_n$, where each U_n is open. Let $(X, d) = (\mathbb{R}$, usual metric $d)$. In each case, explain whether the set S is a G_δ.
(a). $S = \mathbb{Z}$.
(b). $S =$ all irrationals.
(c). $S = \emptyset$.
(d). $S = \mathbb{Q}$.
(e). $S =$ any closed set.

2. Suppose that $\mathcal{F}$ is some set of continuous functions $f : \mathbb{R} \to \mathbb{R}$, having the following property: For each $t \in \mathbb{R}$, there is a constant $M_t > 0$ such that for every $f \in \mathcal{F}, |f(t)| \leq M_t$. Prove that there is a constant M and an open interval $I \subset \mathbb{R}$ such that for all $f \in \mathcal{F}$ and all $t \in I,\ |f(t)| \leq M$.

3(*). Is there a sequence (I_j) of *non-empty, disjoint, closed* subintervals of $I = [0, 1]$ such that $I = \cup_{j=1}^{\infty} I_j$?

4. Let $X = \mathbb{R}^2$. For each $p,\ 1 \leq p < \infty$, define a norm on X as follows: For $x = (x_1, x_2) \in X$, set $||(x_1, x_2)||_p \equiv [|x_1|^p + |x_2|^p]^{\frac{1}{p}}$. Let $\overline{B_p}$ be the closed unit ball in this norm, i.e. $\overline{B_p} = \{x = (x_1, x_2)\ :\ ||x||_p \leq 1\}$.

(a). Draw $\overline{B_1}, \overline{B_2}$, and $\overline{B_p}$ where p is a very large number.

Let's define a further norm on X by $||x||_\infty \equiv \max\{|x_1|, |x_2|\}$.
(b). Draw $\overline{B_\infty}$. (If things have worked out correctly, $\overline{B_p}$ should 'tend' to $\overline{B_\infty}$. Can this statement be made precise?)
(c). For each $p \in [1, \infty]$, find all points x of the form $x = (x_1, 1)$ such that $||x||_p = 1$.

5.(a). For $f \in C[0, 1]$ and $p \in [1, \infty)$, define

$$||f||_p \equiv [\int_0^1 |f(t)|^p dt]^{1/p},$$

and

$$||f||_\infty \equiv \max_{\{0 \leq t \leq 1\}} |f(t)|.$$

Let $f(t) = t$ and $g(t) = 3t - 1$. Calculate $||f - g||_1, ||f - g||_2$, and $||f - g||_\infty$.
(b). Provide the details to show that the space $C[0, 1]$ of continuous real-valued functions on $[0, 1]$ is a *normed space* but *not a Banach space*, when normed by $||f|| = \int_0^1 |f(t)|dt$.