Mathematics 415 2000–01
Exercises A
[Due Monday April 2nd, 2001.]

Let $(c_k)_{k\in\mathbb{Z}}$ be a finitely nonzero sequence of complex numbers,

$$P(z) = \sum_{k\in\mathbb{Z}} c_k z^k, \qquad p(\xi) = P\left(e^{-2\pi i\xi}\right) = \sum_{k\in\mathbb{Z}} c_k e^{-2\pi ik\xi}$$

1. Show that

$$\sum_{k \text{ even}} c_k = \sum_{k \text{ odd}} c_k \iff \sum_k (-1)^k c_k = 0 \iff p\left(\frac{1}{2}\right) = 0 \iff P(-1) = 0.$$

2. Show that

$$\sum_{k \text{ even}} kc_k = \sum_{k \text{ odd}} kc_k \iff \sum_k (-1)^k kc_k = 0 \iff p'\left(\frac{1}{2}\right) = 0 \iff P'(-1) = 0.$$

3. Show that we have both $\sum_k (-1)^k c_k = 0$ and $\sum_k (-1)^k kc_k = 0$ if and only if $P(z)$ is of the form

$$P(z) = z^{-k_0}\left(\frac{1+z}{2}\right)^2 Q(z)$$

   for some polynomial $Q(z)$ and some $k_0 \in \mathbb{Z}$.

   Show that in this case $\sum_k c_k = \sqrt{2} \iff Q(1) = \sqrt{2}$.

4. Find all possible sequences $(c_k)_{k\in\mathbb{Z}}$ with only c_0, c_1, c_2, c_3 nonzero which satisfy all the following conditions:

$$\begin{aligned} \sum_k c_k &= \sqrt{2} \\ \sum_k |c_k|^2 &= 1 \\ \sum_k c_k \overline{c_{k-2\ell}} &= 0 \forall \ell \neq 0 \\ \text{and } \sum_k (-1)^k kc_k &= 0. \end{aligned}$$

   [Hint. Show $\sum_k (-1)^k c_k = 0$. Then $P(z) = \left(\frac{1+z}{2}\right)^2 \left(\sqrt{2} + a(z-1)\right)$ for some $a \in \mathbb{C}$. Find a.]