2 Infinite products and existence of compactly supported ϕ

Infinite products 2.1 *Infinite products $\prod_{n=1}^{\infty} a_n$ need to be defined via limits. But we do* ***not*** *simply say that*

$$\prod_{n=1}^{\infty} a_n = \lim_{N\to\infty} \prod_{n=1}^{N} a_n$$

whenever the limit exists. (For series we do simply do that and define $\sum_{n=1}^{\infty} t_n = \lim_{N\to\infty} \sum_{n=1}^{N} t_n$.)

This overly simple definition of products would allow (for example) $\prod_{n=1}^{\infty}(n-1)$ to converge. It represents

$$0 \times 1 \times 2 \times 3 \times 4 \times \cdots$$

and all the partial products $\prod_{n=1}^{N}(n-1) = 0$. But

$$\prod_{n=2}^{\infty}(n-1) = 1 \times 2 \times 3 \times 4 \times \cdots$$

does not converge — no finite limit for

$$\prod_{n=2}^{N}(n-1) = 1 \times 2 \times \cdots \times (N-1) = (N-1)! \to \infty$$

Definition 2.2 *An infinite product $\prod_{n=1}^{\infty} a_n$ (where $a_n \in \mathbb{C} \forall n$) is said to* converge *if there are at most finitely many n with $a_n = 0$, say $n_1, n_2, \ldots, n_k$, and*

$$\lim_{N\to\infty} \prod_{n=n_k+1}^{N} a_n$$

exists (as a finite limit in $\mathbb{C}$) and is nonzero.

(The same definition works for $a_n \in \mathbb{R}$).

Example 2.3

$$\prod_{n=1}^{\infty} \frac{1}{n}$$

has partial products

$$\prod_{n=1}^{N} \frac{1}{n} = \frac{1}{N!} \to 0 \text{ as } N \to \infty$$

and so this product is not convergent.

Proposition 2.4 *If $\prod_{n=1}^{\infty} a_n$ converges, then*

$$\lim_{n\to\infty} a_n = 1$$

Proof. Let us say that all the zero terms a_n have $n \leq n_0$. Then

$$\lim_{n\to\infty} \prod_{n=n_0+1}^{N} a_n = p \text{ (say) with } p \neq 0$$

Also

$$\lim_{n\to\infty} \prod_{n=n_0+1}^{N+1} a_n = p$$

and dividing the latter by the former we have

$$1 = \frac{p}{p} = \lim_{n\to\infty} \frac{\prod_{n=n_0+1}^{N+1} a_n}{\prod_{n=n_0+1}^{N} a_n} = \lim_{N\to\infty} a_{N+1}$$

Theorem 2.5 *Assuming $a_n \geq 1$ for all n*

$$\prod_{n=1}^{\infty} a_n \textit{ converges} \iff \sum_{n=1}^{\infty} (a_n - 1) < \infty$$

Proof. Note that

$$\prod_{n=1}^{N} a_n = P_N \text{ (say)}$$

is an increasing sequence of positive numbers and so it has a finite limit if and only if it is bounded above. Put $S_N = \sum_{n=1}^{N} (a_n - 1)$. S_N is also increasing with N (as $a_n \geq 1$) and again $\lim_{N\to\infty} S_N \in \mathbb{R}$ if and only if S_N is bounded above.

Observe that $1 + x \leq e^x$ for all $x \geq 0$ and so

$$\begin{aligned}
1 + (a_n - 1) &\leq e^{a_n - 1} \\
a_n &\leq e^{a_n - 1} \\
\prod_{n=1}^{N} a_n &\leq \prod_{n=1}^{N} e^{a_n - 1} = e^{\sum_{n=1}^{N} (a_n - 1)} \\
P_N &\leq e^{S_N}
\end{aligned}$$

But we also have

$$\begin{aligned}
P_N &= \prod_{n=1}^{N} a_n \\
&= \prod_{n=1}^{N} (1 + (a_n - 1)) \\
&\geq 1 + \sum_{n=1}^{N} (a_n - 1) \\
&= 1 + S_N
\end{aligned}$$

Hence

$$1 + S_N \leq P_N \leq e^{S_N}$$

Thus $(S_N)_{N=1}^{\infty}$ is bounded above if and only if $(P_N)_{N=1}^{\infty}$ is bounded above and therefore $\lim_{N\to\infty} S_N < \infty \iff \lim_{N\to\infty} P_N < \infty$.

Definition 2.6 *An infinite product* $\prod_{n=1}^{\infty} a_n$ *is called* absolutely convergent *if*

$$\prod_{n=1}^{\infty} (1 + |a_n - 1|)$$

is convergent.

Lemma 2.7 *An infinite product* $\prod_{n=1}^{\infty} a_n$ *is convergent if and only if it obeys Cauchy's criterion for convergence of products:*

for each $\varepsilon > 0$ *there exists* N_0 *so that*

$$N_2 \geq N_1 \geq N_0 \Rightarrow \left| \prod_{n=N_1}^{N_2} a_n - 1 \right| < \varepsilon$$

Proof. Write

$$\Pi_{\ell,m} = \prod_{n=\ell}^{m} a_n \qquad (m \geq \ell)$$

Assume the product is convergent. We know that there exists $n_0 \geq 1$ and $p \neq 0$ so that

$$\lim_{N\to\infty} \Pi_{n_0,N} = p$$

Given $\varepsilon > 0$, there exists $N_0 > 0$ so that

$$N \geq N_0 \Rightarrow |\Pi_{n_0,N} - p| < \min\left(\frac{\varepsilon|p|}{2}, \frac{|p|}{2}\right) \tag{1}$$

For $N \geq N_0$, we have then

$$\left|\frac{1}{p}\Pi_{n_0,N} - 1\right| < \min\left(\frac{\varepsilon}{2}, \frac{1}{2}\right).$$

Since $|z-1| < 1/2 \Rightarrow 1/2 < |z| \Rightarrow \frac{1}{|z|} < 2$, we deduce that

$$\left|\frac{1}{(1/p)\Pi_{n_0,N}}\right| < 2$$

and hence for $N_2 \geq N_1 > N_0$ we get

$$\begin{aligned}
|\Pi_{N_1,N_2} - 1| &= \left|\frac{\Pi_{n_0,N_2}}{\Pi_{n_0,N_1-1}} - 1\right| \\
&= \left|\frac{(1/p)\Pi_{n_0,N_2}}{(1/p)\Pi_{n_0,N_1-1}} - 1\right| \\
&= \left|\frac{(1/p)\Pi_{n_0,N_2} - (1/p)\Pi_{n_0,N_1-1}}{(1/p)\Pi_{n_0,N_1-1}}\right| \\
&= \left|\frac{(1/p)\Pi_{n_0,N_2} - (1/p)\Pi_{n_0,N_1-1}}{(1/p)\Pi_{n_0,N_1-1}}\right| \\
&\leq 2\left|(1/p)\Pi_{n_0,N_2} - (1/p)\Pi_{n_0,N_1-1}\right| \\
&\leq 2\left(|(1/p)\Pi_{n_0,N_2} - 1| + |1 - (1/p)\Pi_{n_0,N_1-1}|\right) \\
&< 2\left(2\frac{\varepsilon}{2}\right) = \varepsilon.
\end{aligned}$$

Thus the Cauchy condition holds.

Conversely, assume that the Cauchy condition holds. Then there is n_0 so that $N_2 \geq N_1 \geq n_0 \Rightarrow |\Pi_{N_1,N_2} - 1| < 1/2$. In particular, for $N > n_0$

$$|\Pi_{n_0,N} - 1| < \frac{1}{2} \tag{2}$$

Note that $|z-1| < \frac{1}{2} \Rightarrow |z| < 3/2 < 2$.

Now, given any $\varepsilon > 0$ we can find n_1 (and we can assume that $n_1 \geq n_0$) so that

$$N_2 \geq N_1 \geq n_1 \Rightarrow |\Pi_{N_1,N_2} - 1| < \frac{\varepsilon}{2}$$

Then

$$\begin{aligned}
|\Pi_{n_0,N_2} - \Pi_{n_0,N_1}| &= |\Pi_{n_0,N_1}|\left|\frac{\Pi_{n_0,N_2}}{\Pi_{n_0,N_1}} - 1\right| \\
&\leq 2\,|\Pi_{N_1+1,N_2} - 1| \\
&< 2\frac{\varepsilon}{2} = \varepsilon
\end{aligned} \tag{3}$$

as long as $N_2 > N_1 > n_1 + 1$ (and it is valid for $N_2 = N_1$ also). This says that the sequence

$$(\Pi_{n_0,N})_{N=n_0}^{\infty}$$

satisfies the usual Cauchy condition for sequences. Hence

$$\lim_{N\to\infty} \Pi_{n_0,N}$$

exists in $\mathbb{C}$. By (2), the limit is not zero. Hence the product converges. □

Theorem 2.8 *Absolutely convergent infinite products $\prod_{n=1}^{\infty} a_n$ are convergent.*

Proof. Let $b_n = a_n - 1$,

$$\Pi_{\ell,m} = \prod_{n=\ell}^{m} a_n = \prod_{n=\ell}^{m} (1 + b_n)$$

and

$$Q_{\ell,m} = \prod_{n=\ell}^{m} (1 + |b_n|) \qquad (m \geq \ell).$$

Note that

$$|\Pi_{\ell,\ell} - 1| = |(1 + b_\ell) - 1| = |b_\ell| = Q_{\ell,\ell} - 1.$$

By induction on m and for ℓ fixed, we can verify that

$$|\Pi_{m,\ell} - 1| \leq Q_{m,\ell} - 1 \qquad (m \geq \ell).$$

Here is the induction step

$$\begin{aligned}
\Pi_{m+1,\ell} - 1 &= (1 + b_{m+1})\Pi_{\ell,m} - 1 \\
&= (1 + b_{m+1})(\Pi_{\ell,m} - 1) + (1 + b_{m+1}) - 1 \\
&= (1 + b_{m+1})(\Pi_{\ell,m} - 1) + b_{m+1} \\
Q_{m+1,\ell} - 1 &= (1 + |b_{m+1}|)(Q_{\ell,m} - 1) + |b_{m+1}| \\
|\Pi_{m+1,\ell} - 1| &\leq (1 + |b_{m+1}|)\, |\Pi_{\ell,m} - 1| + |b_{m+1}| \\
&\leq (1 + |b_{m+1}|)\, (Q_{m,\ell} - 1) + |b_{m+1}| \\
&\quad \text{by the inductive hypothesis} \\
&= Q_{m+1,\ell} - 1
\end{aligned}$$

Thus the fact that $\prod_{n=1}^{\infty}(1 + |b_n|)$ satisfies the Cauchy criterion for products implies the same of $\prod_{n=1}^{\infty}(1 + b_n)$.

The result follows from Lemma 2.7.

Example 2.9

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n^2}\right)$$

converges no matter what value $z \in \mathbb{C}$ is given.

To justify this example, it is sufficient (by Theorem 2.8) to show that the product is absolutely convergent, that is that

$$\prod_{n=1}^{\infty}\left(1+\left|\frac{z}{n^2}\right|\right)$$

converges. But the convergence of this follows from

$$\sum_{n=1}^{\infty}\frac{|z|}{n^2} < \infty$$

and Theorem 2.5.

To deal with infinite products where the terms are functions, we introduce a notion of uniform convergence.

Definition 2.10 *If $f_n\colon S \to \mathbb{C}$ are complex-valued continuous functions defined on some compact set K, then the product*

$$\prod_{n=1}^{\infty}(1+f_n(z))$$

is called uniformly convergent *on K if there exists n_0 so that $f_n(z) \neq -1$ for $n \geq n_0$ and there is a nowhere-vanishing function $g\colon K \to \mathbb{C}\setminus\{0\}$ so that given any $\varepsilon > 0$ there exists $n_1 \geq n_0$ so that*

$$N \geq n_1, z \in K \Rightarrow \left|\prod_{n=n_0}^{N}(1+f_n(z)) - g(z)\right| < \varepsilon$$

and

$$\inf_{z\in K}|g(z)| > 0.$$

Thus, we require not only that (after omitting a finite number of initial terms) the limits of the partial products be uniform but also that the limit be nonzero in a uniform way.

Proposition 2.11 *If $\prod_{n=1}^{\infty}(1+f_n(z))$ is a uniformly convergent product of continuous functions on a compact set K, then the sequence of partial products*

$$\Pi_{1,N}(z) = \prod_{n=1}^{N}(1+f_n(z))$$

converges uniformly on K (as $N \to \infty$) to

$$\Pi(z) = \prod_{n=1}^{\infty}(1+f_n(z))$$

Proof. From the definition we know that there is n_0 and $g\colon K \to \mathbb{C}$ so that

$$\Pi_{n_0,N}(z) = \prod_{n=n_0}^{N}(1+f_n(z))$$

converges uniformly on K to $g(z)$. By compactness of K and continuity of the f_n

$$\sup_{z\in K} |\Pi_{1,n_0-1}(z)| = M < \infty$$

and then it follows easily that

$$\Pi_{1,N}(z) = \Pi_{1,n_0-1}(z)\Pi_{n_0,N}(z) \to \Pi_{1,n_0-1}(z)g(z) = \Pi(z)$$

uniformly on K.

In fact, given $\varepsilon > 0$ we can choose $N_1 > n_0$ so that

$$N \geq N_1, z \in K \Rightarrow |\Pi_{n_0,N}(z) - g(z)| < \frac{\varepsilon}{M+1}$$

and then it follows that for $N \geq N_1, z \in K$ we have

$$|\Pi_{1,N}(z) - \Pi(z)| = |\Pi_{1,n_0-1}(z)||\Pi_{n_0,N}(z) - g(z)| \leq M\frac{\varepsilon}{M+1} < \varepsilon$$

□

The restriction to products of continuous functions on a compact set which we made in Definition 2.10 is used in the proof of the proposition. Without continuity and compactness, we could still have the result of the proposition by insisting only on uniform boundedness of each f_n, but the applications are usually to the case in the definition.

Exercise. Show that if a K is compact (in $\mathbb{C}$, say) and $f_n\colon K \to \mathbb{C}$ is a sequence of continuous functions so that

$$\prod_{n=1}^{\infty}(1+f_n(z))$$

is uniformly convergent on K, then $\Pi(z) = \prod_{n=1}^{\infty}(1 + f_n(z))$ is continuous on K and $\Pi(z) = 0 \iff \exists n$ such that $1 + f_n(z) = 0$.

As in the case of infinite products of constants, we can show that there is a Cauchy criterion for uniform convergence.

Lemma 2.12 *An infinite product $\prod_{n=1}^{\infty}(1+f_n(z))$, with continuous functions $f_n: K \to \mathbb{C}$ on a compact set K, is uniformly convergent if and only if it satisfies the uniform Cauchy condition:*

given any $\varepsilon > 0$ there exist $N_0 \geq 1$ so that

$$N_2 \geq N_1 \geq N_0, z \in K \Rightarrow \left| \prod_{n=N_1}^{N_2} (1 + f_n(z)) - 1 \right| < \epsilon$$

Proof. Copy the proof of Lemma 2.7 almost word for word, except that (1) has to be replaced by

$$N \geq N_0 \Rightarrow |\Pi_{n_0,N} - g(z)| < \min\left(\frac{\varepsilon}{2}, \frac{1}{2}\right) \inf_{w \in K} |g(w)|$$

Theorem 2.13 *For continuous functions $f_n: K \to \mathbb{C}$ on a compact set K, if*

$$\prod_{n=1}^{\infty}(1 + |f_n(z)|)$$

is uniformly convergent on K, then

$$\prod_{n=1}^{\infty}(1 + f_n(z))$$

is uniformly convergent on K.

Proof. Copy the proof of Theorem 2.8.

Corollary 2.14 (M test) *For continuous functions $f_n: K \to \mathbb{C}$ on a compact set K, if $M_n = \sup_{z \in K} |f_n(z)| < \infty$ for all n and*

$$\sum_{n=1}^{\infty} M_n < \infty,$$

then

$$\prod_{n=1}^{\infty}(1 + f_n(z))$$

is uniformly convergent on K.

Proof. From $\sum_{n=1}^{\infty} M_n < \infty$ and Theorem 2.5 we deduce that $\prod_{n=1}^{\infty}(1+M_n)$ converges.

The Cauchy condition for $\prod_{n=1}^{\infty}(1+M_n)$ implies the uniform Cauchy criterion for $\prod_{n=1}^{\infty}(1+|f_n(z)|)$ and the result then follows from Theorem 2.13.

Proposition 2.15 *If $m(\xi)$ is a trigonometric polynomial with $m(0) = 1$, then*

$$\prod_{n=1}^{\infty} m\left(\frac{\xi}{2^n}\right)$$

converges uniformly on finite intervals $[a, b] \subset \mathbb{R}$.

Proof. As trigonometric polynomials are differentiable and periodic with period 1

$$M = \sup_{0 \le \xi \le 1} |m'(\xi)| = \sup_{\xi \in \mathbb{R}} |m'(\xi)| < \infty$$

It follows that

$$|m(\xi_1) - m(\xi_2)| = \left| \int_{\xi=\xi_2}^{\xi_1} m'(\xi)\, d\xi \right| \le M|xi_1 - \xi_2|$$

and in particular that

$$|m(\xi) - 1| = |m(\xi) - m(0)| \le M|\xi|$$

Now for any finite interval $[a, b]$ we can find $R > 0$ so that $[a, b] \subseteq [-R, R]$ and then for $\xi \in [a, b]$ we have

$$\left| m\left(\frac{\xi}{2^n}\right) - 1 \right| \le M \left| \frac{\xi}{2^n} \right| \le \frac{MR}{2^n}$$

As

$$\sum_{n=1}^{\infty} \frac{MR}{2^n} < \infty$$

uniform convergence of the product follows from Corollary 2.14. □

Example 2.16 *If*

$$m(\xi) = \frac{1 + e^{-2\pi i \xi}}{2}$$

then

$$\prod_{n=1}^{\infty} m\left(\frac{\xi}{2^n}\right) = \int_0^1 e^{-2\pi i x \xi}\, dx = \mathcal{F}(\chi_{[0,1)})(\xi) = e^{-\pi i \xi} \frac{\sin \pi \xi}{\pi \xi}$$

Proof. By induction on N one can show that

$$\prod_{n=1}^{N} \frac{1+e^{-2\pi i(\xi/2^n)}}{2} = \frac{1}{2^N} \sum_{j=0}^{2^N-1} e^{-2\pi i j\xi/2^N}$$

which is a Riemann sum for the integral. It is easy to check that the integral is

$$\frac{e^{-2\pi i\xi}-1}{-2\pi i\xi} = e^{-\pi i\xi}\frac{1}{\pi\xi}\frac{e^{\pi i\xi}-e^{-\pi i\xi}}{2i} = e^{-\pi i\xi}\frac{\sin\pi\xi}{\pi\xi}$$

Alternatively, one can sum the above sum as a geometric progression and arrive at the same limit.

Theorem 2.17 *If* $p(\xi) = \sum_{k=k_0}^{k_1} c_k e^{-2\pi i k\xi}$ *is a trigonometric polynomial with* $p(0) = \sqrt{2}$ *and* $|p(\xi)|^2 + |p(\xi+1/2)|^2 \equiv 2$, *then*

$$\hat{\phi}(\xi) = \prod_{n=1}^{\infty} \frac{p(\xi/2^n)}{\sqrt{2}}$$

defines a function $\hat{\phi} \in L^2(\mathbb{R})$.

If in addition $p(\xi) \neq 0$ *for* $|\xi| < 1/4$, *then* $\|\hat{\phi}\|_2 = 1$ *and*

$$\sum_{\ell\in\mathbb{Z}} |\hat{\phi}(\xi+\ell)|^2 \equiv 1$$

for almost all $\xi \in \mathbb{R}$.

Proof. Let $m(\xi) = p(\xi)/\sqrt{2}$. From Proposition 2.15, we know that the product for $\hat{\phi}$ converges uniformly on all finite intervals $[a,b] \subset \mathbb{R}$ and thus $\hat{\phi}$ is a continuous function.

Define a sequence of functions $\hat{\phi}_r$ $(r = 0, 1, 2, \ldots)$ by

$$\hat{\phi}_0(\xi) = \chi_{[-1/2,1/2)}(\xi)$$

and

$$\begin{aligned}
\hat{\phi}_{r+1}(\xi) &= m(\xi/2)\hat{\phi}_r(\xi/2) \\
&= \left(\prod_{j=1}^{r+1} m\left(\frac{\xi}{2^j}\right)\right) \chi_{[-1/2,1/2)}\left(\frac{\xi}{2^{r+1}}\right) \\
&= \chi_{[-2^r,2^r)}(\xi) \prod_{j=1}^{r+1} m\left(\frac{\xi}{2^j}\right)
\end{aligned}$$

We claim first that

$$\Phi_r(\xi) = \sum_{\ell\in\mathbb{Z}} |\hat{\phi}_r(\xi+\ell)|^2$$

satisfies $\Phi_r(\xi) \equiv 1$ for all $r = 0, 1, 2, \ldots$. This is clear for $r = 0$ and for $r \geq 0$, separating even and odd terms, we get

$$\begin{aligned}
\Phi_{r+1}(\xi) &= \sum_{\ell\in\mathbb{Z}} |\hat{\phi}_{r+1}(\xi+2\ell)|^2 + \sum_{\ell\in\mathbb{Z}} |\hat{\phi}_{r+1}(\xi+1+2\ell)|^2 \\
&= \sum_{\ell\in\mathbb{Z}} |m(\xi/2+\ell)|^2 |\hat{\phi}_r(\xi/2+\ell)|^2 \\
&\quad + \sum_{\ell\in\mathbb{Z}} |m(\xi/2+1/2+\ell)|^2 |\hat{\phi}_r(\xi/2+1/2+\ell)|^2 \\
&= |m(\xi/2)|^2 \sum_{\ell\in\mathbb{Z}} |\hat{\phi}_r(\xi/2+\ell)|^2 \\
&\quad + |m(\xi/2+1/2)|^2 \sum_{\ell\in\mathbb{Z}} |\hat{\phi}_r(\xi/2+1/2+\ell)|^2 \\
&\quad \text{since } m \text{ is periodic with period 1} \\
&= |m(\xi/2)|^2 \Phi_r(\xi/2) + |m(\xi/2+1/2)|^2 \Phi_r(\xi/2+1/2) \\
&= |m(\xi/2)|^2 + |m(\xi/2+1/2)|^2 = 1
\end{aligned}$$

(assuming $\Phi_r \equiv 1$ known).

It then follows that $\|\hat{\phi}_r\|_2 = 1$ for each r, because

$$\begin{aligned}
\int_{-\infty}^{\infty} |\hat{\phi}_r(\xi)|^2 \, d\xi &= \int_{-\infty}^{\infty} \sum_{\ell\in\mathbb{Z}} \chi_{[\ell,\ell+1)} |\hat{\phi}_r(\xi)|^2 \, d\xi \\
&= \sum_{\ell\in\mathbb{Z}} \int_{-\infty}^{\infty} \chi_{[\ell,\ell+1)} |\hat{\phi}_r(\xi)|^2 \, d\xi \\
&\quad \text{(by the monotone converegence theorem)} \\
&= \sum_{\ell\in\mathbb{Z}} \int_{\ell}^{\ell+1} |\hat{\phi}_r(\xi)|^2 \, d\xi \\
&= \sum_{\ell\in\mathbb{Z}} \int_0^1 |\hat{\phi}_r(\eta+\ell)|^2 \, d\eta \\
&= \int_0^1 \sum_{\ell\in\mathbb{Z}} |\hat{\phi}_r(\eta+\ell)|^2 \, d\eta \\
&\quad \text{(by the monotone converegence theorem)} \\
&= \int_0^1 \Phi_r(\eta) \, d\eta = 1
\end{aligned}$$

Now we can argue that

$$\int_{-R}^{R} |\hat{\phi}(\xi)|^2 \, d\xi = \lim_{r\to\infty} \int_{-R}^{R} |\hat{\phi}_r(\xi)|^2 \, d\xi$$

by uniform convergence on $[-R, R]$ and the limit is at most 1 because each term is at most 1 (the r^{th} term in the limit is less than $\|\hat{\phi}_r\|_2^2 = 1$). As

$$\int_{-R}^{R} |\hat{\phi}(\xi)|^2 \, d\xi \leq 1$$

for each $R > 0$, it follows that

$$\|\hat{\phi}\|_2^2 = \int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2 \, d\xi \leq 1$$

Assuming now that $p(\xi) \neq 0$ for $|\xi| < 1/4$, we have $\hat{\phi}(\xi) \neq 0$ for $|\xi| < 1/2$. Since $\hat{\phi}$ is continuous, $c = \inf_{|\xi|\leq 1/2} |\hat{\phi}(\xi)| > 0$. Then for $|\xi| < 2^{r-1}$, we have

$$\hat{\phi}(\xi) = \prod_{n=1}^{\infty} \frac{p(\xi/2^n)}{\sqrt{2}} = \prod_{n=1}^{r} \frac{p(\xi/2^n)}{\sqrt{2}} \prod_{n=r+1}^{\infty} \frac{p(\xi/2^n)}{\sqrt{2}} = \left(\prod_{n=1}^{r} \frac{p(\xi/2^n)}{\sqrt{2}} \right) \hat{\phi}(\xi/2^r)$$

and so

$$|\hat{\phi}_r(\xi)| = \left| \prod_{n=1}^{r} \frac{p(\xi/2^n)}{\sqrt{2}} \right| \leq |\hat{\phi}(\xi)|/|\hat{\phi}(\xi/2^r)| \leq \frac{1}{c}|\hat{\phi}(\xi)|$$

for $|\xi| \leq 2^{r-1}$ and also for $|\xi| > 2^{r-1}$ (when $\hat{\phi}_r(\xi) = 0$).

As we now have $\int_{-\infty}^{\infty} (1/c)^2 |\hat{\phi}(\xi)|^2 \, d\xi \leq (1/c)^2 < \infty$ and $|\hat{\phi}_r(\xi)|^2 \leq (1/c)^2 |\hat{\phi}(\xi)|^2$, we can apply the dominated convergence theorem to show

$$\int_{-\infty}^{\infty} \lim_{r\to\infty} |\hat{\phi}_r(\xi)|^2 \, d\xi = \lim_{r\to\infty} \int_{-\infty}^{\infty} |\hat{\phi}_r(\xi)|^2 \, d\xi = 1$$

which tells us

$$\int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2 \, d\xi = 1$$

Finally, we have

$$\begin{aligned} \sum_{\ell=-N}^{N} |\hat{\phi}(\xi+\ell)|^2 &= \lim_{r\to\infty} \sum_{\ell=-N}^{N} |\hat{\phi}_r(\xi+\ell)|^2 \\ &\leq \limsup_{r\to\infty} \sum_{\ell=-\infty}^{\infty} |\hat{\phi}_r(\xi+\ell)|^2 \\ &= \limsup_{r\to\infty} \Phi_r(\xi) = 1 \end{aligned}$$

and hence

$$\sum_{\ell=-\infty}^{\infty} |\hat{\phi}(\xi+\ell)|^2 \leq 1.$$

But then

$$1 = \int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2 \, d\xi = \int_0^1 \sum_{\ell=-\infty}^{\infty} |\hat{\phi}(\xi+\ell)|^2 \leq \int_0^1 1 \, d\xi = 1$$

and so $\sum_{\ell=-\infty}^{\infty} |\hat{\phi}(\xi+\ell)|^2 = 1$ for almost all $\xi \in [0,1]$. As the sum is periodic with period 1, and a countable union of sets of measure zero still has measure zero, the sum is 1 for all $\xi \in \mathbb{R}$ except possibly for a set of measure zero. $\square$

Corollary 2.18 *If $(c_k)_{k\in\mathbb{Z}}$ is a finitely nonzero sequence with*

$$\sum_{k\in\mathbb{Z}} c_k \overline{c_{k-2\ell}} = \begin{cases} 1 & \text{for } \ell = 0 \\ 0 & \text{for } \ell \in \mathbb{Z}, \ell \neq 0. \end{cases}$$

$\sum_k c_k = \sqrt{2}$ and $\sum_k c_k e^{-2\pi i k\xi} \neq 0$ for $|\xi| < 1/4$, then there is a function $\phi \in L^2(\mathbb{R})$ with $\|\phi\|_2 = 1$,

$$\phi = \sum_{k\in\mathbb{Z}} c_k D_2 T_k \phi$$

and orthonormal translates $\{T_k\phi : k \in \mathbb{Z}\}$.

Proof. We know from Lemma 1.13 that $p(\xi) = \sum_k c_k e^{-2\pi i k\xi}$ satisfies $|p(\xi)|^2 + |p(\xi+1/2)|^2 \equiv 2$ and also we have $p(0) = \sum_k c_k = \sqrt{2}$. From Theorem 2.17, we have

$$\hat{\phi}(\xi) = \prod_{n=1}^{\infty} \frac{p(\xi)}{\sqrt{2}} \in L^2(\mathbb{R})$$

with $\|\hat{\phi}\|_2 = 1$. If we define ϕ to be the inverse Fourier transform of $\hat{\phi}$, then we have $\|\phi\|_2 = \|\hat{\phi}\|_2 = 1$ and

$$\begin{aligned}
\mathcal{F}\phi(\xi) &= \hat{\phi}(\xi) \\
&= p(\xi/2)\frac{1}{\sqrt{2}}\hat{\phi}(\xi/2) \\
&= p(\xi/2)\frac{1}{\sqrt{2}}\mathcal{F}\phi(\xi/2) \\
&= \mathcal{F}\left(\sum_{k\in\mathbb{Z}} c_k D_2 T_k \phi\right)(\xi)
\end{aligned}$$

Since $\mathcal{F}$ is bijective on $L^2(\mathbb{R})$, we can conclude that ϕ satisfies the dilation equation.

Finally, for $j \neq k$

$$\begin{aligned}
\langle T_j\phi, T_k\phi\rangle &= \langle T_{-j}T_j\phi, T_{-j}T_k\phi\rangle \\
&\quad \text{since } T_j \text{ is an isometry} \\
&= \langle \phi, T_{k-j}\phi\rangle \\
&= \langle \mathcal{F}\phi, \mathcal{F}(T_{k-j}\phi)\rangle \\
&\quad \text{since the Fourier transform is an isometry on } L^2(\mathbb{R}) \\
&= \int_{-\infty}^{\infty} \hat{\phi}(\xi)\overline{e^{2\pi i(k-j)\xi}\hat{\phi}(\xi)}\, d\xi \\
&= \int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2 e^{-2\pi i(k-j)\xi}\, d\xi \\
&= \sum_{\ell\in\mathbb{Z}} \int_{\ell}^{\ell+1} |\hat{\phi}(\xi)|^2 e^{-2\pi i(k-j)\xi}\, d\xi \\
&= \sum_{\ell\in\mathbb{Z}} \int_0^1 |\hat{\phi}(\xi+\ell)|^2 e^{-2\pi i(k-j)\xi}\, d\xi \\
&= \int_0^1 \sum_{\ell\in\mathbb{Z}} |\hat{\phi}(\xi+\ell)|^2 e^{-2\pi i(k-j)\xi}\, d\xi \\
&\quad \text{by the dominated convergence theorem, since} \\
&\quad\quad \sum_{\ell\in\mathbb{Z}} |\hat{\phi}(\xi+\ell)|^2 = 1 \text{ ae} \\
&= \int_0^1 e^{-2\pi i(k-j)\xi}\, d\xi \\
&= 0
\end{aligned}$$

□

Our next aim is to show that the solution ϕ of the dilation equation that we have constructed is compactly supported (and hence in $L^1(\mathbb{R})$ as well as $L^2(\mathbb{R})$). There are a number of ways to do this directly, but we will take an approach that relies on a classical result in Fourier analysis that we will not prove.

By a compactly supported function in $L^2(\mathbb{R})$ we mean one that is almost everywhere zero outside some bounded closed interval $[a, b]$.

Recall that an *entire function* g is a complex-valued function defined and holomorphic (or analytic) on the whole complex plane $\mathbb{C}$. That means that

$$g'(w_0) = \lim_{w\to w_0} \frac{g(w) - g(w_0)}{w - w_0}$$

exists in $\mathbb{C}$ for each $w_0 \in \mathbb{C}$.

An entire function is said to be of *exponential type* if there are constants $A, B > 0$ so that

$$|g(w)| \leq Ae^{B|w|}$$

holds for all $w \in \mathbb{C}$. Examples of such functions are $g(w) = \cos w = (e^{iw} - e^{-iw})/2$ and polynomial functions $g(w) = \sum_{j=0}^{n} a_j w^j$ (which satisfy $|g(w)| \leq a(1 + |w|)^n \leq Ae^{B|w|}$ for a suitable $a > 0$ and any $B > 0$ with a suitable A).

Theorem 2.19 (Paley-Wiener) *A function $f \in L^2(\mathbb{R})$ is compactly supported if and only if its Fourier transform $\mathcal{F}f(\xi)$ is almost everywhere equal for $\xi \in \mathbb{R}$ to $g(\xi)$ for an entire function $g \colon \mathbb{C} \to \mathbb{C}$ of exponential type.*

Proof. Omitted.

Theorem 2.20 *If $(c_k)_{k\in\mathbb{Z}}$ is a finitely nonzero sequence with*

$$\sum_{k\in\mathbb{Z}} c_k \overline{c_{k-2\ell}} = \begin{cases} 1 & \text{for } \ell = 0 \\ 0 & \text{for } \ell \in \mathbb{Z}, \ell \neq 0. \end{cases}$$

$\sum_k c_k = \sqrt{2}$ and $\sum_k c_k e^{-2\pi i k\xi} \neq 0$ for $|\xi| < 1/4$, then there is a compactly supported function $\phi \in L^2(\mathbb{R})$ with $\|\phi\|_2 = 1$ and

$$\phi = \sum_{k\in\mathbb{Z}} c_k D_2 T_k \phi$$

Proof. What we do is show that the Fourier transform of the ϕ constructed in the proof of Corollary 2.18 above satisfies the conditions in Theorem 2.19.

Let $m(w) = (1/\sqrt{2}) \sum_k c_k e^{-2\pi i k w}$ (with $w \in \mathbb{C}$ now replacing $\xi \in \mathbb{R}$) and

$$g(w) = \prod_{n=1}^{\infty} m\left(\frac{w}{2^n}\right)$$

A look at the proof of Proposition 2.15 will show that we did not really make use of the fact that ξ was real in any essential way. The same argument shows that the product for $g(w)$ is uniformly convergent for $|w| \leq R$, any $R > 0$. Thus $g \colon \mathbb{C} \to \mathbb{C}$ is well defined and continuous. In fact it is also entire because of a standard result in complex analysis that if a sequence of entire functions on $\mathbb{C}$ converges uniformly on compact subsets of $\mathbb{C}$, then the limit is also entire. (In this case the partial products $\prod_{n=1}^{N} m(w/2^n)$ are clearly entire.) We have $\mathcal{F}\phi(\xi) = g(\xi)$ for $\xi \in \mathbb{R}$ and what remains is to establish the exponential type of g.

For $|w| < 1$ we know there is a constant $M \geq 0$ so that $|m(w) - 1| \leq M|w|$ and so

$$\begin{aligned} |g(w)| &= \left| \prod_{n=1}^{\infty} m\left(\frac{w}{2^n}\right) \right| \\ &\leq \prod_{n=1}^{\infty} (1 + |m(w/2^n) - 1|) \\ &\leq \prod_{n=1}^{\infty} (1 + M/2^n) = M' < \infty \end{aligned}$$

holds for $|w| < 2$. Now for $|w| \geq 2$, choose N so that $2^N \leq |w| < 2^{N+1}$. Then

$$g(w) = \left(\prod_{n=1}^{N} m\left(\frac{w}{2^n}\right) \right) g\left(\frac{w}{2^N}\right)$$

and we need a bound on this finite product in terms of a power of $e^{|w|}$.

Suppose c_k is zero outside the (finite) range $k_0 \leq k \leq k_1$. Consider for a moment the polynomial

$$P(z) = (1/\sqrt{2}) \sum_{k=k_0}^{k_1} c_k z^{k-k_0}$$

so that

$$m(w) = e^{-2\pi i k_0 w} P(e^{-2\pi i w})$$

Now, being a polynomial of degree $d = k_1 - k_0$ we can find $C > 0$ so that

$$|P(z)| \leq C|z|^d \text{ for } |z| \geq 1$$

and $|P(z)| \leq C$ for $|z| < 1$. Thus

$$|m(w)| \leq e^{2\pi |k_0||w|} C (e^{2\pi |w|})^d = C e^{2\pi (d + |k_0|)|w|}$$

holds for all $w \in \mathbb{C}$ (as $|e^w| = e^{\Re w} \leq e^{|w|}$ and $e^{|w|} \geq 1$ always).

It follows that for $2^N \leq |w| < 2^{N+1}$

$$|g(w)| \leq \left(\prod_{n=1}^{N} C e^{2\pi(d+|k_0|)|w|/2^n} \right) \left| g\left(\frac{w}{2^{N+1}}\right) \right| \leq M' C^N e^{2\pi(d+|k_0|)|w|}$$

As $N < \log|w| / \log 2$ we have $C^N = e^{N \log C} < e^{(\log C/\log 2)\log|w|} \leq e^{(\log C/\log 2)|w|}$ (since $|w| \geq 1$). Thus for we have

$$|g(w)| \leq M' e^{(\log C/\log 2 + 2\pi(d+|k_0|))|w|}$$

no matter what N is. Taking $A = M'$ and $B = \max(\log C/\log 2 + 2\pi(d + |k_0|), 0)$ we have

$$|g(w)| \leq Ae^{B|w|}$$

for all $w \in \mathbb{C}$. $\square$

Theorem 2.21 *Suppose $(c_k)_{k\in\mathbb{Z}}$ is a finitely nonzero sequence with $c_k = 0$ for k outside the range $k_0 \leq k \leq k_1$,*

$$\sum_{k\in\mathbb{Z}} c_k \overline{c_{k-2\ell}} = \begin{cases} 1 & \text{for } \ell = 0 \\ 0 & \text{for } \ell \in \mathbb{Z}, \ell \neq 0. \end{cases}$$

$\sum_k c_k = \sqrt{2}$, $\sum_{k \text{ even}} kc_k = \sum_{k \text{ odd}} kc_k$, *and* $\sum_k c_k e^{-2\pi i k\xi} \neq 0$ *for* $|\xi| < 1/4$. *Factor*

$$P_1(z) = \sum_{k=k_0}^{k_1} c_k z^{k-k_0} = \sqrt{2}\left(\frac{1+z}{2}\right)^a Q(z)$$

for a polynomial $Q(z)$ with $Q(-1) \neq 0$ and a an integer.

Assume

$$\sup\{|Q(z)| : z \in \mathbb{C}, |z| = 1\} = B < 2^{a-1}$$

Then there is a compactly supported continuous function $\phi \in L^2(\mathbb{R})$ with $\|\phi\|_2 = 1$ and

$$\phi = \sum_{k\in\mathbb{Z}} c_k D_2 T_k \phi$$

Proof. From Lemma 1.13, we know that $p(\xi) = e^{-2\pi i k_0 \xi} P_1(e^{-2\pi i\xi})$ satisfies $|p(\xi)|^2 + |p(\xi + 1/2)|^2 = 2$, $p(0) = P_1(1) = \sqrt{2}$ and consequently $p(1/2) = P_1(-1) = 0$. The condition $\sum_{k \text{ even}} kc_k = \sum_{k \text{ odd}} kc_k$ guarantees that the polynomial $P_1(z)$ has derivative 0 at -1.

We conclude that $P_1(z)$ is divisible by $(1 + z)^2$. Let a be the highest power of $1 + z$ which divides $P_1(z)$ and then we can factor $P_1(z)$ as in the statement (with $a \geq 2$).

Our goal is to show that $\hat{\phi}(\xi)$ as constructed in Theorem 2.17 must be in $L^1(\mathbb{R})$. It follows that the inverse Fourier transform of $\hat{\phi}$ is a continuous function

$$\phi(x) = \int_{-\infty}^{\infty} \hat{\phi}(\xi) e^{2\pi i x\xi}\, dx$$

and we already know from the proof of Corollary 2.18 that this satisfies the dilation equation.

Take

$$m(\xi) = p(\xi)/\sqrt{2} = e^{-2\pi i k_0 \xi}\left(\frac{1 + e^{-2\pi i\xi}}{2}\right)^a Q(e^{-2\pi i\xi})$$

and, as in the proof of Theorem 2.17, $\hat{\phi}(\xi) = \prod_{n=1}^{\infty} m(\xi/2^n)$. From Proposition 2.15, we know that each of the three products

$$\prod_{n=1}^{\infty} e^{-2\pi i k_0 \xi/2^n}, \prod_{n=1}^{\infty} \left(\frac{1 + e^{-2\pi i \xi/2^n}}{2} \right) \text{ and } \prod_{n=1}^{\infty} Q(e^{-2\pi i \xi/2^n})$$

converges. The first product has constant modulus 1, the second has been computed in Example 2.16 to be $e^{-\pi i \xi} \frac{\sin \pi\xi}{\pi\xi}$. It is easy to see that the infinite product of a product is the product of the individual infinite products provided each of the individual products converges. Thus

$$\hat{\phi}(\xi) = \left(\prod_{n=1}^{\infty} e^{-2\pi i k_0 \xi/2^n} \right) \left(e^{-\pi i \xi} \frac{\sin \pi\xi}{\pi\xi} \right)^a \prod_{n=1}^{\infty} Q(e^{-2\pi i \xi/2^n})$$

To achieve our goal of showing $\int_{-\infty}^{\infty} |\hat{\phi}(\xi)| \, d\xi < \infty$ we need an estimate on the last product.

For convenience, let $q(\xi) = Q(e^{-2\pi i \xi})$, a trigonometric polynomial. For $-1 \leq \xi \leq 1$ We have an estimate based on the largest value of the derivative $M = \sup_\xi |q'(\xi)|$ that

$$|q(\xi) - 1| = |q(\xi) - q(0)| \leq M|\xi|$$

and so for $|\xi| \leq 1$ we have

$$\left| \prod_{n=1}^{\infty} q(\xi/2^n) \right| \leq \prod_{n=1}^{\infty} (1 + |q(\xi/2^n) - 1|) \leq \prod_{n=1}^{\infty} (1 + M/2^n) < e^M$$

For $|\xi| > 1$ we argue in a somewhat similar way to the previous proof. Fix ξ. Choose $N \geq 0$ so that $2^{N-1} < |\xi| \leq 2^N$. Then

$$\left| \prod_{n=1}^{\infty} q(\xi/2^n) \right| = \left| \prod_{n=1}^{N} q(\xi/2^n) \right| \left| \prod_{n=N+1}^{\infty} q(\xi/2^n) \right| \leq \left(\prod_{n=1}^{N} |q(\xi/2^n)| \right) e^M$$

But each $|q(\xi/2^n)| = |Q(e^{-2\pi i \xi/2^n})| \leq B$ and so

$$\left| \prod_{n=1}^{\infty} q(\xi/2^n) \right| \leq B^N e^M$$

But $2^N < 2|\xi|$ and so $B^N e^M < e^M 2^{N \log_2 B} < e^M (2|\xi|)^{\log_2 B}$. We conclude that, for $|\xi| > 1$

$$|\hat{\phi}(\xi)| \leq \left| \frac{1}{\pi\xi} \right|^a e^M (2|\xi|)^{\log_2 B} = C \frac{1}{|\xi|^{a - \log_2 B}}$$

for some C. As we have the assumption $a - \log_2 B > 1$, it follows that

$$\int_{|\xi|>1} |\hat{\phi}(\xi)|\, d\xi < \infty$$

and $\int_{-1}^{1} |\hat{\phi}(\xi)|\, d\xi$ is also finite (bounded by $\int_{-1}^{1} \left|\frac{\sin \pi\xi}{\pi\xi}\right|^a e^M\, d\xi < \infty$).

Thus $\hat{\phi} \in L^1(\mathbb{R})$. $\square$

Remark 2.22 *Theorem 2.21 can be used to show that there exist compactly supported continuous solutions (with orthonormal translates) to certain dilation equations.*

Daubechies produced a family of examples for $a = 2, 3, \ldots$. *The corresponding* $p_a(\xi) = \sum_{k=0}^{2a-1} c_k e^{-2\pi i k\xi}$ *is of the form*

$$p_a(\xi) = \sqrt{2}\left(\frac{1 + e^{-2\pi i\xi}}{2}\right)^a q_a(\xi)$$

with $q_a(\xi)$ *a trigonometric polynomial that is chosen in such a way that*

$$|p_a(\xi)|^2 + |p_a(\xi + 1/2)|^2 = 2.$$

Now

$$\begin{aligned} |p_a(\xi)|^2 &= p_a(\xi)\overline{p_a(\xi)} \\ &= 2\left(\frac{1 + e^{-2\pi i\xi}}{2}\right)^a \left(\frac{1 + e^{2\pi i\xi}}{2}\right)^a |q_a(\xi)|^2 \\ &= 2\left(\frac{2 + e^{2\pi i\xi} + e^{-2\pi i\xi}}{4}\right)^a |q_a(\xi)|^2 \\ &= 2\left(\frac{1 + \cos(2\pi\xi)}{2}\right)^a |q_a(\xi)|^2 \\ &= 2\left(\frac{1 + 1 - 2\sin^2(\pi\xi)}{2}\right)^a |q_a(\xi)|^2 \\ &= 2(1 - \sin^2(\pi\xi))^a |q_a(\xi)|^2 \end{aligned}$$

and Daubechies chose q_a *so that*

$$|q_a(\xi)|^2 = \sum_{k=0}^{a-1} \binom{a+k-1}{k} (\sin^2(\pi\xi))^k$$

Using

$$\sin^2(\pi(\xi + 1/2)) = \sin^2(\pi\xi + \pi/2) = \cos^2(\pi\xi) = 1 - \sin^2(\pi\xi)$$

we can show

$$\begin{aligned}
&|p_a(\xi)|^2 + |p_a(\xi+1/2)|^2 \\
= \;& 2(1-\sin^2(\pi\xi))^a \sum_{k=0}^{a-1} \left(\begin{array}{c} a+k-1 \\ k \end{array} \right) (\sin^2(\pi\xi))^k \\
& +2(\sin^2(\pi\xi))^a \sum_{k=0}^{a-1} \left(\begin{array}{c} a+k-1 \\ k \end{array} \right) (1-\sin^2(\pi\xi))^k \\
= \;& 2.
\end{aligned}$$

For example in the case $a = 2$, this is true because (with $s = \sin^2(\pi\xi)$) it comes down to

$$\begin{aligned}
&2(1-s)^2(1+2s) + 2s^2(1+2(1-s)) \\
&\quad = 2((1-2s+s^2)(1+2s) + s^2(3-2s)) \\
&\quad = 2(1-4s^2+s^2+2s^3+3s^2-2s^3) \\
&\quad = 2.
\end{aligned}$$

For the Daubechies 4 coefficient example, writing $z = e^{-2\pi i\xi}$, we have

$$\begin{aligned}
p(\xi) &= \frac{1+\sqrt{3}}{4\sqrt{2}} + \frac{3+\sqrt{3}}{4\sqrt{2}}z + \frac{3-\sqrt{3}}{4\sqrt{2}}z^2 + \frac{1-\sqrt{3}}{4\sqrt{2}}z^3 \\
&= \sqrt{2}\left(\frac{1}{8}\right)(1+2z+z^2)((1-\sqrt{3})z + (1-\sqrt{3})) \\
&\qquad \textit{using long division} \\
&= \sqrt{2}\left(\frac{1+z}{2}\right)^2 \left(\frac{(1-\sqrt{3})z + (1+\sqrt{3})}{2}\right)
\end{aligned}$$

and so

$$q(\xi) = Q(z) = \frac{(1-\sqrt{3})z + (1+\sqrt{3})}{2}$$

If we compute

$$\begin{aligned}
|q(\xi)|^2 &= q(\xi)\overline{q(\xi)} \\
&= \frac{1}{4}((1-\sqrt{3})^2 z\overline{z} + (1+\sqrt{3})^2 + (1-3)z + (1-3)\overline{z}) \\
&= \frac{1}{4}((4-2\sqrt{3}) + (4+2\sqrt{2}) - 2(z+\overline{z})) \\
&\qquad \textit{using } |z| = \left|e^{-2\pi i\xi}\right| = 1 \\
&= 2 - \cos(2\pi\xi) \\
&= 2 - (1 - 2\sin^2(\pi\xi)) \\
&= 1 + 2\sin^2(\pi\xi) = 1 + 2s
\end{aligned}$$

(where $s = \sin^2(\pi\xi)$ *still). We can see then that*

$$B = \sup\{|q_a(\xi)| : \xi \in \mathbb{R}\} = \sqrt{\sup_{0 \le s \le 1} (1 + 2s)} = \sqrt{3} < 2^{a-1} = 2^{2-1} = 2$$

and by Theorem 2.21, the corresponding ϕ *is continuous. So is the mother wavelet* ψ *constructed from* ϕ*.*

We should check all the hypotheses of Theorem 2.21.