1 Multiresolution Analysis Approach to Wavelets

(This is probably the most practical approach to wavelets in $L^2(\mathbb{R})$, but not the only approach.)

Lemma 1.1 *The translation and dilation operators*

$$\begin{aligned} T_a \colon L^2(\mathbb{R}) &\to L^2(\mathbb{R}) \\ (T_a(f))(x) &= f(x-a) \qquad (a \in \mathbb{R}) \\ D_\lambda \colon L^2(\mathbb{R}) &\to L^2(\mathbb{R}) \\ (D_\lambda(f))(x) &= \sqrt{\lambda} f(\lambda x) \qquad (\lambda > 0) \end{aligned}$$

are isometries, that is

$$\begin{aligned} \|T_a f\|_2 &= \|f\|_2 \\ \|D_\lambda f\|_2 &= \|f\|_2 \qquad \forall f \in L^2(\mathbb{R}). \end{aligned}$$

Consequently, they preserve inner products:

$$\begin{aligned} \langle T_a f, T_a g\rangle &= \langle f, g\rangle \\ \langle D_\lambda f, D_\lambda g\rangle &= \langle f, g\rangle \qquad \forall f, g \in L^2(\mathbb{R}). \end{aligned}$$

Proof. Simple calculations show that these operators are isometries:

$$\begin{aligned} \|T_a f\|_2^2 &= \int_{-\infty}^{\infty} |T_a f(x)|^2\, dx \\ &= \int_{-\infty}^{\infty} |f(x-a)|^2\, dx \\ &= \int_{-\infty}^{\infty} |f(y)|^2\, dy \\ &= \|f\|_2^2 \\ \|D_\lambda f\|_2^2 &= \int_{-\infty}^{\infty} |D_\lambda f(x)|^2\, dx \\ &= \int_{-\infty}^{\infty} \lambda |f(\lambda x)|^2\, dx \\ &= \int_{-\infty}^{\infty} |f(y)|^2\, dy \qquad (y = \lambda x) \\ &= \|f\|_2^2 \end{aligned}$$

The inner product preserving consequence follows from the polarisation formula in $L^2(\mathbb{R})$. For $\mathbb{R}$-valued $L^2(\mathbb{R})$, this is

$$\begin{aligned}\langle f, g\rangle &= \frac{1}{4}(\|f+g\|_2^2 - \|f-g\|_2^2)\\ &= \frac{1}{4}(\langle f+g, f+g\rangle - \langle f-g, f-g\rangle)\end{aligned}$$

and for $\mathbb{C}$-valued $L^2(\mathbb{R})$, this gives the real part $\Re\langle f, g\rangle$, so that

$$\begin{aligned}\langle f, g\rangle &= \Re\langle f, g\rangle + i\Im\langle f, g\rangle\\ &= \Re\langle f, g\rangle + i\Re\langle f, ig\rangle\\ &\quad \text{since}\\ &\quad \langle f, ig\rangle = -i\langle f, g\rangle\\ &\quad \Re\langle f, ig\rangle = \Im\langle f, g\rangle\\ \langle f, g\rangle &= \frac{1}{4}(\|f+g\|_2^2 - \|f-g\|_2^2 + i\|f+ig\|_2^2 - i\|f-ig\|_2^2)\end{aligned}$$

From these formulae it follows easily that if $T\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is linear and isometric, then it is automatically inner product preserving. For the (shorter) $\mathbb{R}$-valued case here are the details:

$$\begin{aligned}\langle Tf, Tg\rangle &= \Re\langle Tf, Tg\rangle + i\Im\langle Tf, Tg\rangle\\ &= \frac{1}{4}(\|Tf+Tg\|_2^2 - \|Tf-Tg\|_2^2)\\ &= \frac{1}{4}(\|T(f+g)\|_2^2 - \|T(f-g)\|_2^2)\\ &= \frac{1}{4}(\|f+g\|_2^2 - \|f-g\|_2^2)\\ &\quad \text{since } \|Th\| = \|h\| \quad \forall h\\ &= \langle f, g\rangle\end{aligned}$$

Since it is easy to check that T_a and D_λ are linear, this completes the proof. □

Lemma 1.2

$$D_\lambda T_a = T_{a/\lambda} D_\lambda$$

Proof. Write $g = T_a(f)$ so that $g(x) = f(x-a)$ and

$$(D_\lambda T_a f)(x) = (D_\lambda g)(x) = \sqrt{\lambda}g(\lambda x) = \sqrt{\lambda}f(\lambda x - a)$$

One the other hand, write $h = D_\lambda f$ so that $h(x) = \sqrt{\lambda} f(\lambda x)$ and

$$(T_{a/\lambda} D_\lambda f)(x) = (T_{a/\lambda} h)(x) = h(x - a/\lambda) = \sqrt{\lambda} f(\lambda(x - a/\lambda)) = f(\lambda x - a)$$

$\square$

Starting point for construction of wavelets 1.3 *A function $\phi(x)$ (generally known as a* ***scaling function****) with the properties*

(i) The translates $(T_k\phi)_{k\in\mathbb{Z}}$ are orthonormal inside $L^2(\mathbb{R})$

(ii) ϕ satisfies a two scale dilation equation

$$\begin{aligned} \phi(x) &= \sum_{k\in\mathbb{Z}} c_k \sqrt{2}\phi(2x - k) \\ &= \sum_{k\in\mathbb{Z}} c_k (D_2 T_k \phi)(x) \end{aligned}$$

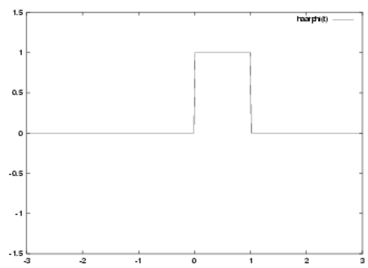
(Scale 1 on the left, scale 2 on the right, x and $2x$.)

Normally we assume that only a finite number of the c_k are nonzero, but in theory we can can allow any sequence of coefficients with $\sum_{k\in\mathbb{Z}} |c_k|^2 = 1$.

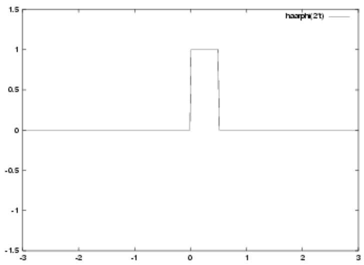
Example 1.4 *A simple example is*

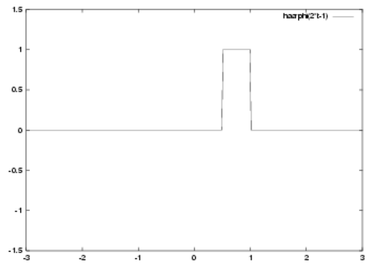
$$\begin{aligned} \phi(x) &= \chi_{[0,1)}(x) \\ &= \begin{cases} 1 & \text{if } x \in [0,1) \\ 0 & \text{if } x \notin [0,1) \end{cases} \end{aligned}$$

Graph of this is



The graphs of $\phi(2x) = \chi_{[0,1/2)}(x)$ and $\phi(2x - 1) = \phi(2(x - 1/2)) = \chi_{[1/2,1)}(x)$ are shown next.





Hence $\phi(x) = \phi(2x) + \phi(2x - 1)$ *or* $\chi_{[0,1)}(x) = \chi_{[0,1/2)}(x) + \chi_{[1/2,1)}(x)$ *or*

$$\phi(x) = c_0\sqrt{2}\phi(2x) + c_1\sqrt{2}\phi(2x - 1) \text{ with } c_0 = c_1 = \frac{1}{\sqrt{2}}$$

Take $c_k = 0$ *for* $k \notin \{0, 1\}$ *and we have*

$$\phi(x) = \sum_{k\in\mathbb{Z}} c_k\sqrt{2}\phi(2x - k)$$

This ϕ related to Haar wavelets because the basic Haar function

$$\psi(x) = \begin{cases} 1 & x \in [0, 1/2) \\ -1 & x \in [1/2, 1) \\ 0 & \text{for } x < 0 \text{ and for } x \geq 1 \end{cases}$$

can be expressed as

$$\begin{aligned} \psi(x) &= \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x) \\ &= c_1\sqrt{2}\phi(2x) - c_0\sqrt{2}\phi(2x - 1) \end{aligned}$$

In general, we will be able to take any scaling function ϕ (which has to have some additional properties) and get a ψ (basic wavelet) by introducing alternating signs (and a reversed order) for the coefficients of the dilation equation.

Proposition 1.5 (Properties required of the c_k's) *Assuming all the time that* $\phi = \sum_{k\in\mathbb{Z}} c_k D_2 T_k \phi$ *and* $T_k\phi$ *are orthonormal in* $L^2(\mathbb{R})$, *we have*

(i)

$$\sum_{k\in\mathbb{Z}} |c_k|^2 = 1$$

(ii)

$$\sum_{k\in\mathbb{Z}} c_k\overline{c_{k-2\ell}} = 0 \text{ for } \ell \in \mathbb{Z}, \ell \neq 0.$$

(iii) Assuming $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $\int_{-\infty}^{\infty} \phi(x)\,dx \neq 0$ *and* $\{k \in \mathbb{Z} : c_k \neq 0\}$ *is finite, then we can write a formula for the Fourier transform of* ϕ*:*

$$\mathcal{F}\phi(\xi) = \mathcal{F}\phi(0) \lim_{n\to\infty} \prod_{j=1}^{n} \frac{1}{\sqrt{2}} p\left(\frac{\xi}{2^j}\right)$$

where $p(\xi)$ *is the trigonometric polynomial*

$$p(\xi) = \sum_{k\in\mathbb{Z}} c_k e^{-2\pi i k\xi}$$

(iv) If ϕ *is compactly supported (that is there is a bounded set of* $x \in \mathbb{R}$ *where almost all* x *with* $\phi(x) \neq 0$ *are to be found) then* $\phi \in L^1(\mathbb{R})$*.*

If in addition $\{k : c_k \neq 0\}$ *is finite and* $\int_{-\infty}^{\infty} \phi(x)\,dx \neq 0$*, then*

$$\sum_{k\in\mathbb{Z}} c_k = \sqrt{2}$$

Proof.

(i) The dilation equation reads

$$\phi(x) = \sum_{k\in\mathbb{Z}} c_k \sqrt{2}\phi(2x - k) = \sum_{k\in\mathbb{Z}} c_k (D_2 T_k \phi)(x)$$

Since we are assuming that $(T_k\phi)_{k\in\mathbb{Z}}$ are orthonormal in $L^2(\mathbb{R})$, we have $\|T_k\phi\|_2 = 1$ (all k) and $\langle T_k\phi, T_\ell\phi\rangle = 0$ if $k \neq \ell$. Since D_2 is isometric (and inner product preserving — see Lemma 1.1),

$$(D_2 T_k \phi)_{k\in\mathbb{Z}}$$

is orthonormal and so we can calculate by Besssels formula

$$\|\phi\| = \sqrt{\sum_{k\in\mathbb{Z}} |c_k|^2}$$

As $\|\phi\| = 1$ (because $\phi = T_0\phi$ is in the sequence $(T_k\phi)_{k\in\mathbb{Z}}$ which is assumed to be orthonormal), we have

$$\sum_{k\in\mathbb{Z}} |c_k|^2 = 1.$$

(ii) Starting from the observation that $\phi = T_0\phi$ is perpendicular to $T_\ell\phi$ for $\ell \neq 0$, we compute from $\langle \phi, T_\ell\phi\rangle = 0$ by expressing

$$\begin{aligned}
\phi &= \sum_{k\in\mathbb{Z}} c_k D_2 T_k \phi \\
T_\ell \phi &= \sum_{k\in\mathbb{Z}} c_k T_\ell D_2 T_k \phi \\
&\quad \text{justified even for infinitely many nonzero } c_k \\
&\quad \text{as } T_\ell \text{ is continuous and linear} \\
&= \sum_{k\in\mathbb{Z}} c_k D_2 T_{2\ell} T_k \phi \\
&= \sum_{k\in\mathbb{Z}} c_k D_2 T_{k+2\ell} \phi \\
&= \sum_{k\in\mathbb{Z}} c_{k-2\ell} D_2 T_k \phi \\
0 = \langle \phi, T_\ell \phi\rangle &= \sum_{k\in\mathbb{Z}} c_k \overline{c_{k-2\ell}}
\end{aligned}$$

(iii) Considering

$$\phi = \sum_{k\in\mathbb{Z}} c_k D_2 T_k \phi$$

and applying $\mathcal{F}$ to both sides, we get

$$\mathcal{F}\phi = \sum_{k\in\mathbb{Z}} c_k \mathcal{F}(D_2 T_k \phi).$$

(To justify this for the case of r infinitely many nonzero c_k we can use the fact that $\mathcal{F}$ is an isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$.) Now we need to know rules $(\mathcal{F}(D_2 f))(\xi) = (D_{1/2}(\mathcal{F}f))(\xi)$ and $(\mathcal{F}(T_k f))(\xi) = e^{-2\pi i k\xi}(\mathcal{F}f)(\xi)$.

Aside. We now check out these claimed formulae by simple calculations with integrals and changes of variable. We take $f \in L^1(\mathbb{R})$ so that the integral formulae are valid without question.

$$\begin{aligned}
(\mathcal{F}(D_2 f))(\xi) &= \int_{-\infty}^{\infty} (D_2 f)(x) e^{-2\pi i x\xi}\, dx \\
&= \int_{-\infty}^{\infty} \sqrt{2} f(2x) e^{-2\pi i x\xi}\, dx \\
&\quad \text{put } y = 2x
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \sqrt{2} f(y) e^{-2\pi i y(\xi/2)}\,\frac{dy}{2} \\
&= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} f(y) e^{-2\pi i y(\xi/2)}\,dy \\
&= \frac{1}{\sqrt{2}} (\mathcal{F} f)(\xi) \\
&= (D_{1/2}(\mathcal{F} f))(\xi) \\
(\mathcal{F}(T_k f))(\xi) &= \int_{-\infty}^{\infty} (T_k f)(x) e^{-2\pi i x\xi}\,dx \\
&= \int_{-\infty}^{\infty} f(x-k) e^{-2\pi i x\xi}\,dx \\
&\quad \text{put } y = x-k \qquad dy = dx \\
&= \int_{-\infty}^{\infty} f(y) e^{-2\pi i (y+k)\xi}\,dy \\
&= e^{-2\pi k\xi} \int_{-\infty}^{\infty} f(y) e^{-2\pi i y\xi}\,dy \\
&= e^{-2\pi k\xi} (\mathcal{F} f)(\xi)
\end{aligned}
$$

[Aside within aside: We could introduce a notation

$$(R_\theta f)(x) = e^{-2\pi i\theta x} f(x)$$

and then we could summarise the rule we have just proved as $\mathcal{F}(T_k f) = R_k(\mathcal{F} f)$, but we will not use this notation regularly.]

Returning now to the proof proper, we have

$$
\begin{aligned}
(\mathcal{F}\phi)(\xi) &= \sum_{k\in\mathbb{Z}} c_k (\mathcal{F}(D_2 T_k \phi))(\xi) \\
&= \sum_{k\in\mathbb{Z}} c_k (D_{1/2}(\mathcal{F}(T_k\phi)))(\xi) \\
&= \sum_{k\in\mathbb{Z}} c_k (D_{1/2}(e^{-2\pi i k\xi} (\mathcal{F}\phi)(\xi)) \\
&= \sum_{k\in\mathbb{Z}} c_k \frac{1}{\sqrt{2}} e^{-2\pi i k(\xi/2)} (\mathcal{F}\phi)(\xi/2)) \\
&= \left(\sum_{k\in\mathbb{Z}} c_k e^{-2\pi i k(\xi/2)} \right) \frac{1}{\sqrt{2}} (\mathcal{F}\phi)\left(\frac{\xi}{2}\right) \\
(\mathcal{F}\phi)(\xi) &= p\left(\frac{\xi}{2}\right) \frac{1}{\sqrt{2}} (\mathcal{F}\phi)\left(\frac{\xi}{2}\right)
\end{aligned}
$$

If we rewrite this with ξ replaces by $\xi/2$ we get

$$(\mathcal{F}\phi)\left(\frac{\xi}{2}\right) = p\left(\frac{\xi}{4}\right)\frac{1}{\sqrt{2}}(\mathcal{F}\phi)\left(\frac{\xi}{4}\right)$$

and combining the last two equations we then have

$$(\mathcal{F}\phi)(\xi) = p\left(\frac{\xi}{2}\right)\frac{1}{\sqrt{2}}p\left(\frac{\xi}{4}\right)\frac{1}{\sqrt{2}}(\mathcal{F}\phi)\left(\frac{\xi}{4}\right).$$

Iterating this idea, we get

$$(\mathcal{F}\phi)(\xi) = \left(\prod_{j=1}^{n}\frac{1}{\sqrt{2}}p\left(\frac{\xi}{2^j}\right)\right)(\mathcal{F}\phi)\left(\frac{\xi}{2^n}\right) \qquad (n = 1, 2, 3, \ldots).$$

Take limits as $n \to \infty$. As $\phi \in L^1(\mathbb{R})$ we know that $\mathcal{F}\phi$ is continuous, in particular continuous at 0 and so

$$\lim_{n\to\infty}(\mathcal{F}\phi)\left(\frac{\xi}{2^n}\right) = (\mathcal{F}\phi)(0) \neq 0.$$

It follows that

$$\lim_{n\to\infty}\prod_{j=1}^{n}\frac{1}{\sqrt{2}}p\left(\frac{\xi}{2^j}\right) = \frac{(\mathcal{F}\phi)(\xi)}{(\mathcal{F}\phi)(0)}$$

exists.

[**Aside:** When $(\mathcal{F}\phi)(\xi) \neq 0$ this can be rewritten as

$$\prod_{j=1}^{\infty}\frac{1}{\sqrt{2}}p\left(\frac{\xi}{2^j}\right) = \frac{(\mathcal{F}\phi)(\xi)}{(\mathcal{F}\phi)(0)}$$

but we will defer going into the definition of an infinite product.]

Hence we have

$$(\mathcal{F}\phi)(\xi) = (\mathcal{F}\phi)(0)\lim_{n\to\infty}\prod_{j=1}^{n}\frac{1}{\sqrt{2}}p\left(\frac{\xi}{2^j}\right).$$

Note that $(\mathcal{F}\phi)(0) = \int_{-\infty}^{\infty}\phi(x)\,dx$.

(iv) For the first part suppose that $\phi(x)$ is essentially supported in $[a, b]$, by which we means that $\{x \notin [a, b] : \phi(x) \neq 0\}$ has measure zero. Then

$$\begin{aligned}
\|\phi\|_1 &= \int_{-\infty}^{\infty} |\phi(x)|\, dx \\
&= \int_a^b |\phi(x)|\, dx \\
&\le \sqrt{\int_a^b |\phi(x)|^2\, dx}\sqrt{\int_a^b 1^2\, dx} \\
&\qquad \text{by Hölder's inequality} \\
&= \|\phi\|_2 \sqrt{b-a} < \infty
\end{aligned}$$

For the second part of (iv), we integrate both sides of

$$\begin{aligned}
\phi(x) &= \sum_{k=-\infty}^{\infty} c_k \sqrt{2} \phi(2x-k) \\
\int_{-\infty}^{\infty} \phi(x)\, dx &= \sum_{k=-\infty}^{\infty} c_k \sqrt{2} \int_{-\infty}^{\infty} \phi(2x-k)\, dx \\
&\qquad \text{noting that the exchange of the sum and the integral is justified because there are only a finite number of nonzero terms in the sum} \\
&\qquad \text{put } y = 2x - k \\
&\qquad\quad dy = 2\, dx \\
&\qquad \text{in the integrals} \\
&= \sum_{k=-\infty}^{\infty} c_k \sqrt{2} \int_{-\infty}^{\infty} \phi(y)\, \frac{dy}{2} \\
&= \left(\sum_{k=-\infty}^{\infty} c_k \frac{\sqrt{2}}{2} \right) \int_{-\infty}^{\infty} \phi(y)\, dy
\end{aligned}$$

Since we have assumed $\int_{-\infty}^{\infty} \phi(x)\, dx \neq 0$, we find

$$\sum_{k=-\infty}^{\infty} c_k \frac{1}{\sqrt{2}} = 1$$

or

$$\sum_{k=-\infty}^{\infty} c_k = \sqrt{2}.$$

Remarks. We have one example where all the conditions we have now uncovered on the coefficients c_k are satisfied. For $\phi(x) = \chi_{[0,1)}(x)$, $c_0 = c_1 = 1/\sqrt{2}$, $c_k = 0 \forall k \neq 0$ or 1.

$$\begin{aligned} c_0 + c_1 &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \\ |c_0|^2 + |c_1|^2 &= \frac{1}{2} + \frac{1}{2} \end{aligned}$$

$$\begin{array}{lllllllll} c_k & \ldots, & c_{-1} = 0, & c_0, & c_1, & c_2 = 0, & \ldots & & \\ c_{k-2} & \ldots, & & & c_{-1} = 0, & c_0, & c_1, & c_2 = 0 & \\ & & & \uparrow & & \uparrow & & & \\ & & & k = 0 & & k = 2 & & & \end{array}$$

$$\sum_k c_k \overline{c_{k-2\ell}} = 0 \text{ for } \ell \neq 0.$$

In fact there are no other sequences c_k with only c_0 and c_1 nonzero that meet all these conditions. To check this take two number c_0 and c_1 and suppose we know $c_1 + c_2 = \sqrt{2}$ and $|c_0|^2 + |c_1|^2 = 1$. Then from the Cauchy Schwarz inequality

$$\begin{aligned} \sqrt{2} = |c_0 + c_1| &= |c_0 \times 1 + c_1 \times 1| \\ &\leq \sqrt{|c_0|^2 + |c_1|^2}\sqrt{1^1 + 1^2} \\ &= 1\sqrt{2} \end{aligned}$$

Thus equality holds in Cauchy Schwarz and so (c_0, c_1) is linearly dependent on $(1, 1)$. That is $(c_0, c_1) = c_0(1, 1)$ and $c_0 = c_1$. Since $c_0 + c_1 = \sqrt{2}$ we must have $c_0 = c_1 = 1/\sqrt{2}$.

If we allow 3 consecutive terms c_0, c_1, c_2 to be nonzero, then the orthogonality condition $\sum_k \overline{c_{k-2}} = 0$ comes down to $c_2\overline{c_0} = 0$ and so either $c_0 = 0$ or $c_2 = 0$. This means we are back to two nonzero consecutive terms and having $c_1 = c_2 = 1/\sqrt{2}$ is not essentially different from the Haar case. (The scaling function ϕ in that case is $\chi_{[1,2)}$.)

Daubechies example. If we allow 4 nonzero terms c_0, c_1, c_2, c_3, then the solution

$$c_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}, c_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}, c_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}, c_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$$

was used by I. Daubechies in 1988. The corresponding $\phi(x)$ is continuous and compactly supported.

We can easily check that these numbers satisfy the constraints we have identified:

$$
\begin{aligned}
&|c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 \\
&\quad = \frac{(1+\sqrt{3})^2}{32} + \frac{(3+\sqrt{3})^2}{32} + \frac{(3-\sqrt{3})^2}{32} + \frac{(1-\sqrt{3})^2}{32} \\
&\quad = \frac{1+2\sqrt{3}+3+9+6\sqrt{3}+3}{32} \\
&\qquad + \frac{9-6\sqrt{3}+3+1-2\sqrt{3}+3}{32} \\
&\quad = \frac{32}{32} = 1
\end{aligned}
$$

$$
\begin{aligned}
c_0 + c_1 + c_2 + c_3 &= \frac{1+\sqrt{3}+3+\sqrt{3}+3-\sqrt{3}+1-\sqrt{3}}{4\sqrt{2}} \\
&= \frac{8}{4\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}
\end{aligned}
$$

The orthogonality relation boils down to

$$
\begin{array}{cccccc}
c_0 & c_1 & c_2 & c_3 & & \\
 & & \overline{c_0} & \overline{c_1} & \overline{c_2} & \overline{c_2}
\end{array}
$$

$$
\begin{aligned}
c_2\overline{c_0} + c_3\overline{c_1} &= \left(\frac{3-\sqrt{3}}{4\sqrt{2}}\right)\left(\frac{1+\sqrt{3}}{4\sqrt{2}}\right) + \left(\frac{1-\sqrt{3}}{4\sqrt{2}}\right)\left(\frac{3+\sqrt{3}}{4\sqrt{2}}\right) \\
&= \frac{1}{4\sqrt{2}}(3-\sqrt{3}+3\sqrt{3}-3+3-3\sqrt{3}+\sqrt{3}-3) \\
&= 0
\end{aligned}
$$

Although we have now checked that this sequence satisfies all the necessary conditions we have uncovered so far that are necessary for the existence of a compactly supported $L^2(\mathbb{R})$ solution with $\int_{-\infty}^{\infty} \phi(x)\, dx \neq 0$ of a dilation equation $\phi = \sum_k c_k D_2 T_k \phi$, we still don't have proof that there is any such ϕ with the 4 Daubechies coefficients above.

Lemma 1.6 *Suppose the finite dilation equation*

$$\phi(x) = \sum_{k=k_0}^{k_1} c_k \sqrt{2} \phi(2x-k)$$

has a compactly supported solution valid for all $x \in \mathbb{R}$ (pointwise). Then

$$\{x : \phi(x) \neq 0\} \subseteq [k_0, k_1]$$

Proof. Consider $\phi(x)$ restricted to intervals $[\ell, \ell+1)$ where $\ell \in \mathbb{Z}$. There can only be a finite number of ℓ where $\phi(x)$ is not identically zero on $[\ell, \ell+1)$ (since the support is compact). Look at the smallest such ℓ.

We *claim* that $\ell \geq k_0$.

For $x \in [\ell, \ell+1)$, we know

$$\phi(x) = \sum_{k=k_0}^{k_1} c_k \sqrt{2}\phi(2x-k)$$

but when we look at where $2x-k$ is we see that

$$2\ell - k \leq 2x - k < 2(\ell+1) - k.$$

For $k \geq k_0$

$$\begin{aligned} 2(\ell+1) - k \leq 2(\ell+1) - k_0 &= \ell + \ell + 2 - k_0 \\ &= \ell + (\ell - k_0 + 1) + 1 \end{aligned}$$

Suppose now that $\ell < k_0$ contrary to what we claimed (and then we will try to get a contradiction). Then $\ell \leq k_0 - 1$ (since $\ell, k_0 \in \mathbb{Z}$) and so $\ell - k_0 + 1 \leq 0$. Hence

$$2x - k < \ell + 1$$

and for $k \geq k_0 + 1$ we have

$$2x - k < 2(\ell+1) - k \leq \ell \Rightarrow \phi(2x-k) = 0.$$

This the only term on the right hand side of the dilation equation that can survive (for $x \in [\ell, \ell+1)$) is the first term. Thus

$$\phi(x) = c_{k_0}\sqrt{2}\phi(2x - k_0) \text{ for } x \in [\ell, \ell+1)$$

Looking now at $x \in [\ell, \ell + 1/2)$ we have

$$2x - k_0 < 2(\ell + 1/2) - k_0 = 2\ell + 1 - k_0 = \ell + (\ell + 1 - k_0) \leq \ell$$

Thus $\phi(2x - k_0) = 0$ and so

$$\phi(x) = c_{k_0}\sqrt{2}\phi(2x - k_0) \text{ for } x \in [\ell, \ell + 1/2)$$

We can then use this to show that for $x \in [\ell, \ell + 3/4)$

$$2x - k_0 < 2(\ell + 3/4) - k_0 = 2\ell + 3/2 - k_0 = \ell + 1/2 + (\ell + 1 - k_0) \leq \ell + 1/2$$

Thus $\phi(2x - k_0) = 0$ and so

$$\phi(x) = c_{k_0}\sqrt{2}\phi(2x - k_0) \text{ for } x \in [\ell, \ell + 3/4)$$

By induction we can show that

$$\phi(x) = 0 \text{ for } x \in [\ell, \ell + 1 - 1/2^n) \qquad (n = 1, 2, \dots)$$

Since $\bigcup_{n=1}^{\infty}[\ell, \ell + 1 - 1/2^n)[\ell, \ell + 1)$ we conclude that $\phi(x) = 0$ must be true for all $x \in [\ell, \ell + 1)$. That contradicts the choose of ℓ and shows that $\ell < k_0$ is impossible.

We have shown

$$\{x : \phi(x) \neq 0\} \subseteq [k_0, \infty).$$

To show that

$$\{x : \phi(x) \neq 0\} \subseteq (-\infty, k_0]$$

we could use a similar argument again, but we could instead note that the "reflected" function

$$\phi^r(x) = \phi(-x)$$

satisfies

$$\begin{aligned}
\phi^r(x) = \phi(-x) &= \sum_{k=k_0}^{k_1} c_k\sqrt{2}\phi(2(-x) - k) \\
&= \sum_{k=k_0}^{k_1} c_k\sqrt{2}\phi(-2x - k) \\
&= \sum_{k=k_0}^{k_1} c_k\sqrt{2}\phi^r(2x + k) \\
&\quad \text{substitute } k = -j \\
&= \sum_{j=-k_1}^{-k_0} c_j\sqrt{2}\phi^r(2x - j)
\end{aligned}$$

According to the first part of the proof, we have

$$\{x : \phi^r(x) \neq 0\} \subseteq [-k_1, \infty) \Rightarrow \{x : \phi(x) \neq 0\} \subseteq (-\infty, k_1]$$

This completes the proof that ϕ is supported in $[k_0, k_1]$. $\square$

Method for graphing ϕ 1.7 *The assumptions needed now are:*

1. *ϕ compactly supported*

2. *ϕ continuous (a new assumption)*

3. *ϕ satisfies a finite dilation equation*

$$\phi(x) = \sum_{k=k_0}^{k_1} c_k \sqrt{2}\phi(2x-k)$$

(By taking $T_{-k_0}\phi$ in place of ϕ we could concentrate on the case $k_0 = 0$ but this is not essential.)

Step 1. Find the sequence of values of ϕ at the integers,

$$(\phi(n))_{n=-\infty}^{\infty} = (\ldots, \phi(k_0), \phi(k_0+1), \ldots, \phi(k_1), \ldots)$$

Since $\phi(x) = 0$ for $x < k_0$ and for $x > k_1$ and ϕ is assumed continuous, we must have $\phi(k_0) = 0$ and $\phi(k_1) = 0$. That leaves

$$(\phi(k_0+1), \phi(k_0+2), \ldots, \phi(k_1-1))$$

We know

$$\phi(n) = \sum_{k=k_0}^{k_1} c_k \sqrt{2}\phi(2n-k).$$

Note that $2n - k \in \mathbb{Z}$ and and we can express these equations as a single matrix equation

$$\begin{pmatrix} \phi(k_0+1) \\ \phi(k_0+2) \\ \vdots \\ \phi(k_1-1) \end{pmatrix} = \begin{pmatrix} \text{matrix with} \\ c\text{'s as entries} \end{pmatrix} \begin{pmatrix} \phi(k_0+1) \\ \phi(k_0+2) \\ \vdots \\ \phi(k_1-1) \end{pmatrix}$$

Now we have

$$\begin{aligned} \phi(k_0+1) &= c_{k_0}\sqrt{2}\phi(2k_0+2-k_0) + c_{k_0+1}\sqrt{2}\phi(2k_0+2-k_0-1) \\ &\quad + c_{k_0+2}\sqrt{2}\phi(2k_0+2-k_0-2) + \cdots \\ &= c_{k_0}\sqrt{2}\phi(k_0+2) + c_{k_0+1}\sqrt{2}\phi(k_0+1) + 0 + 0 + \cdots \end{aligned}$$

and this means that the first row of the above matrix is

$$(c_{k_0+1}\sqrt{2}, c_{k_0}\sqrt{2}, 0, 0, \ldots)).$$

The second row turns out to be

$$(c_{k_0+3}\sqrt{2}, c_{k_0+2}\sqrt{2}, c_{k_0+1}\sqrt{2}, c_{k_0}\sqrt{2}, 0, 0, \ldots)).$$

In this way we can show that the matrix above has rows made up of the c's running backwards (times $\sqrt{2}$) and each successive row is shifted by 2 places.

$$\begin{pmatrix} \phi(k_0+1) \\ \phi(k_0+2) \\ \vdots \\ \phi(k_1-1) \end{pmatrix}$$
$$= \begin{pmatrix} c_{k_0+1}\sqrt{2} & c_{k_0}\sqrt{2} & 0 & 0 & \dots \\ c_{k_0+3}\sqrt{2} & c_{k_0+2}\sqrt{2} & c_{k_0+1}\sqrt{2} & c_{k_0}\sqrt{2} & 0 \\ \vdots & & & & \end{pmatrix} \begin{pmatrix} \phi(k_0+1) \\ \phi(k_0+2) \\ \vdots \\ \phi(k_1-1) \end{pmatrix}$$

Thus we have an eigenvector with eigenvalue 1 for the above matrix.

If the eigenspace is 1-dimensional, this is enough to find

$$(\phi(k_0+1), \phi(k_0+2), \dots, \phi(k_1-1))$$

up to a scale factor (if the eigenspace is one dimensional).

Step 2. Next we use the dilation equation at a 1/2 integer $x = j + 1/2$

$$\phi(x) = \phi(j+1/2) = \sum_{k=k_0}^{k_1} c_k\sqrt{2}\phi(2j+1-k)$$

and we see that the right hand side uses only values of ϕ at integers $2j+1-k$ (which we found at step 1).

Once we have ϕ at 1/2 integers, we can use the dilation equation with $x = j + 1/4$ and $x = j + 3/4$. For example

$$\phi(j+1/4) = \sum_{k=k_0}^{k_1} c_k\sqrt{2}\phi(2j+1/2-k)$$

and the right hand side involves only values at 1/2 integers.

In this way we can find

$$\phi\left(\frac{j}{2^n}\right) \qquad j \in \mathbb{Z}, \quad n = 1, 2, 3, \dots$$

Example 1.8 We can carry out this procedure for the Daubechies example mentioned earlier.

$$c_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}, c_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}, c_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}, c_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$$

Here $k_0 = 0$ and $k_1 = 3$. The nonzero values at integers are $\phi(k_0 + 1), \ldots, \phi(k_1 - 1)$ which means just the two values $(\phi(1), \phi(2))$ in this case.

From the above we know that $\begin{pmatrix} \phi(1) \\ \phi(2) \end{pmatrix}$ must be an eigenvector with eigenvalue 1 for the matrix

$$\begin{pmatrix} c_1\sqrt{2} & c_0\sqrt{2} \\ c_3\sqrt{2} & c_2\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{3+\sqrt{3}}{4} & \frac{1+\sqrt{3}}{4} \\ \frac{1-\sqrt{3}}{4} & \frac{3-\sqrt{3}}{4} \end{pmatrix}$$

To find the 1-eigenspace, subtract I_2 and look for the kernel of

$$\begin{pmatrix} \frac{3+\sqrt{3}}{4} - 1 & \frac{1+\sqrt{3}}{4} \\ \frac{1-\sqrt{3}}{4} & \frac{3-\sqrt{3}}{4} - 1 \end{pmatrix} = \begin{pmatrix} \frac{-1+\sqrt{3}}{4} & \frac{1+\sqrt{3}}{4} \\ \frac{1-\sqrt{3}}{4} & \frac{-1-\sqrt{3}}{4} \end{pmatrix}$$

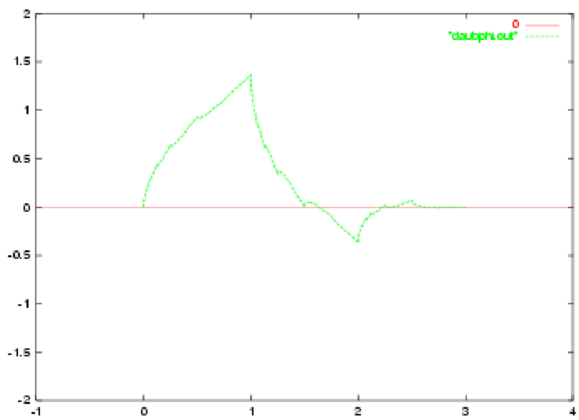
A vector in the kernel is

$$\begin{pmatrix} \sqrt{3}+1 \\ 1-\sqrt{3} \end{pmatrix}$$

and so we must have $\begin{pmatrix} \phi(1) \\ \phi(2) \end{pmatrix}$ some multiple of $\begin{pmatrix} \sqrt{3}+1 \\ 1-\sqrt{3} \end{pmatrix}$.

One thing that we could do is to take the multiple so that $\phi(1)+\phi(2) = 1$ (that would mean the multiple should be $1/2$ of the above eigenvector).

If we then use the procedure outlined above to write a computer programme, we can find values for ϕ and then plot it.



Here is a computer programme written in `perl` that produces lines with coordinates of points on the graph (which can then be plotted with `gnuplot` or another graphical utility).

```
#!/usr/bin/perl

%phi;
%c;

$c{0} = (1 + sqrt(3))/4;
$c{1} = (3 + sqrt(3))/4;
$c{2} = (3 - sqrt(3))/4;
$c{3} = (1 - sqrt(3))/4;

$maxlevel = 5;

$phi{0} = 0;
$phi{3} = 0;

$phi{1} = (1 + sqrt(3))/2;
$phi{2} = (1 - sqrt(3))/2;

$step = 1;

foreach my $lev (0..$maxlevel) {
        $step = 1/2**$lev;
        $base = $step/2;
        foreach my $k (1..(3*2**$lev)) {
            $x = $base + ($k-1)*$step;
            $phi{$x} = 0;
            foreach my $j (0..3) {
                $phi{$x} = $phi{$x} + $c{$j}*$phi{2*$x - $j};
            }
        }
}

foreach my $k (0..(3*(2**$maxlevel))) {
        $x = $k*$step;
        print "$x $phi{$x}\n";
}
```

Exercise. Show that if ϕ is a compactly supported continuous function and it satisfies a 2-scale dilation equation

$$\phi = \sum_{k=k_0}^{k_1} c_k D_2 T_k \phi$$

and if $\sum_{n\in\mathbb{Z}} \phi(n) = 1$, then $\sum_{n\in\mathbb{Z}} \phi\left(\frac{n}{2^j}\right) = 2^j$ for $j = 1, 2, \ldots$. If also $\int_{-\infty}^{\infty} \phi(x)\,dx \neq 0$, then show that $\int_{-\infty}^{\infty} \phi(x)\,dx = 1$.

Lemma 1.9 *Suppose ϕ is a compactly supported continuous solution of a finite 2-scale dilation equation*

$$\phi = \sum_{k=k_0}^{k_1} c_k D_2 T_k \phi$$

and suppose

$$f(x) = \sum_{\ell} a_\ell \phi(x - \ell)$$

is a finite linear combination of integer translates of ϕ.

Then f is completely determined by its values $(f(n))_{n=-\infty}^{\infty}$ at the integers.

Note. We saw that ϕ is determined by its values at the integers.

One way to express the above result is to say that f is determined by its samples with spacing 1.

Proof. (of Lemma 1.9) We want to show we can find the coefficients a_ℓ by knowing only $f(n)$ for all n. Or another way to put it is that if we have a second finite linear combination

$$g(x) = \sum_{\ell} \tilde{a}_\ell \phi(x - \ell)$$

and $g(n) = f(n)$ for all $n \in \mathbb{Z}$, then $f(x) = g(x)$ for all x.

Looking at $f(x) - g(x)$, this amounts to showing that if $f(n) = 0$ for each n, then $a_\ell = 0$ for all ℓ (and so $f(x) \equiv 0$).

So, now suppose that $f(n) = 0$ for all $n \in \mathbb{Z}$ but $f \not\equiv 0$ so that some $a_\ell \neq 0$.

We know ϕ has compact support and so there are only finitely many k with $\phi(k) \neq 0$. From the graphing procedure above, we know that if $\phi(k) = 0$ for all k, then $\phi \equiv 0$. Of course, the case $\phi \equiv 0$ is trivial as certainly $f \equiv 0$ then. (In fact this proof uses only two properties of ϕ: compact support and some $k \in \mathbb{Z}$ with $\phi(k) \neq 0$.)

Write $k_{\min}$ for the smallest k with $\phi(k) \neq 0$, and $k_{\max}$ for the largest. Thus the nonzero values at integers $\phi(n)$ are among

$$\phi(k_{\min}) \neq 0, \phi(k_{\min}+1), \ldots, \phi(k_{\max}-1), \phi(k_{\max}) \neq 0$$

Now

$$\begin{aligned} f(n) &= \sum_{\ell} a_\ell \phi(n-\ell) \\ &\qquad k_{\min} \leq n-\ell \leq k_{\max} \\ &\qquad \Rightarrow -k_{\min} \geq n-\ell \geq -k_{\max} \\ &= \sum_{\ell=n-k_{\max}}^{n-k_{\min}} a_\ell \phi(n-\ell). \end{aligned}$$

Choose n so that $n - k_{\min} =$ smallest ℓ with $a_\ell \neq 0$. Call this $\ell_{\min}$.

Then

$$\begin{aligned} f(n) &= \sum_{\ell=n-k_{\max}}^{n-k_{\min}} a_\ell \phi(n-\ell) \\ &= a_{n-k_{\min}} \phi(n-(n-k_{\min})) \\ &= a_{\ell_{\min}} \phi(k_{\min}) \end{aligned}$$

As we are assuming that $f(n) = 0$ for all n, and $\phi(k_{\min}) \neq 0$, we conclude from this that $a_{\ell_{\min}} = 0$. But that contradicts the way $\ell_{\min}$ is chosen.

This contradiction shows that $a_\ell = 0$ for all ℓ if $f(n) = 0$ for all n, and completes the proof. □

Construction of a basic wavelet.

$$\psi(x) = \sum_{k\in\mathbb{Z}} (-1)^k \overline{c_{1-k}} \sqrt{2} \phi(2x-k)$$

assuming that $\phi(x)$ satisfies a two scale dilation equation

$$\phi(x) = \sum_{k\in\mathbb{Z}} c_k \sqrt{2} \phi(2x-k)$$

and ϕ has orthonormal translates $T_k\phi$.

Lemma 1.10 (Helps to graph ψ) *If ψ satisfies a finite dilation equation*

$$\phi = \sum_{k=k_0}^{k_1} c_k D_2 T_k \phi$$

and ϕ is continuous and compactly supported, then ψ as constructed above is nonzero only between

$$x = \frac{k_0 + 1 - k_1}{2} \text{ and } x = \frac{k_1 + 1 - k_0}{2}.$$

Note. For the Daubechies 4 coefficient example, we have $k_0 = 0$, $k_1 = 3$ and ψ is then supported in the interval

$$\left[\frac{k_0 + 1 - k_1}{2}, \frac{k_1 + 1 - k_0}{2}\right] = \left[\frac{0 + 1 - 3}{2}, \frac{3 + 1 - 0}{2}\right] = [-1, 2].$$

Proof. (of Lemma 1.10) From Lemma 1.6 we know that ϕ is supported in $[k_0, k_1]$ and so if $\phi(2x - k) \neq 0$, then $k_0 \leq 2x - k \leq k_1$ and thus

$$\frac{k_0 + k}{2} \leq x \leq \frac{k_1 + k}{2}.$$

$$\psi(x) = \sum_{k \in \mathbb{Z}} (-1)^k \overline{c_{1-k}} \sqrt{2} \phi(2x - k)$$

Note that $c_{1-k} \neq 0 \Rightarrow k_0 \leq 1 - k \leq k_1 \Rightarrow 1 - k_0 \geq k \geq 1 - k_1$.

If $\psi(x) \neq 0$, then there must be at least one nonzero term in the summation. Thus

$$\phi(2x - k) \neq 0 \text{ for some } 1 - k_1 \leq k \leq 1 - k_0$$

and so

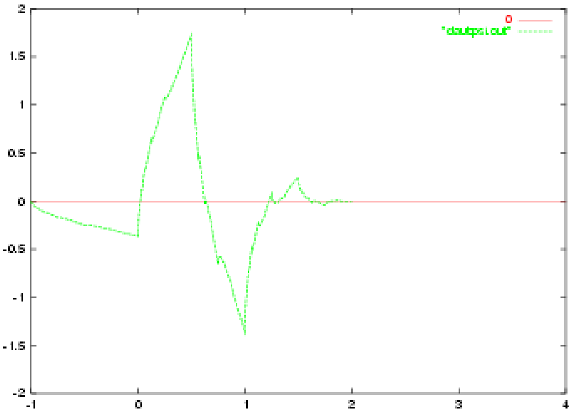
$$\frac{k_0 + k}{2} \leq x \leq \frac{k_1 + k}{2} \text{ for one of these } k.$$

It follows that

$$\frac{k_0 + 1 - k_1}{2} \leq x \leq \frac{k_1 + 1 - k_0}{2}.$$

Observation. The interval $[(k_0+1-k_1)/2, (k_1+1-k_0)/2]$ has length $k_1 - k_0$, the same length as the interval $[k_0, k_1]$ where ϕ is supported.

Assuming that the ϕ in the Daubechies example is compactly supported and continuous, we can use the method above for computing values of ϕ at points $j/2^n$ to deduce values of ψ and so write a computer programme (a slight modification of the earlier one) to plot ψ.



Theorem 1.11 ***[Important properties of*** ψ***]*** *Assume ϕ has orthonormal translates $T_k\phi$ and satisfies $\phi = \sum_{k\in\mathbb{Z}} c_k D_t T_k \phi$. Assume*

$$\psi = \sum_{k\in\mathbb{Z}} (-1)^k \overline{c_{1-k}} D_2 T_k \phi.$$

Then:

(i) *The translates $T_\ell \psi$ ($\ell \in \mathbb{Z}$) are orthonormal.*

(ii) *If we define $V_0 =$ the closed linear span in $L^2(\mathbb{R})$ of $\{T_k\phi : k \in \mathbb{Z}\}$ and $V_1 = D_2 V_0 =$ the closed linear span in $L^2(\mathbb{R})$ of $\{D_2 T_k \phi : k \in \mathbb{Z}\}$, then*

$$V_0 \subseteq V_1,$$

each $T_\ell\psi \in V_1$ and

$$\{T_k\phi : k \in \mathbb{Z}\} \cup \{T_\ell\psi : \ell \in \mathbb{Z}\}$$

is an orthonormal basis of V_1.

(We can say that $\{T_\ell\psi : \ell \in \mathbb{Z}\}$ is an orthonormal basis for the orthogonal complement of V_0 inside V_1.)

(iii)

$$\{D_{2^n} T_\ell \psi : n, \ell \in \mathbb{Z}\}$$

is an orthonormal set in $L^2(\mathbb{R})$.

(iv) *Put $V_n = D_{2^n} V_0$ for $n \in \mathbb{Z}$ (which fits with the above definitions of V_0 and V_1).* If we have

$$\bigcap_{n\in\mathbb{Z}} V_n = \{0\} \text{ and } \bigcup_{n\in\mathbb{Z}} V_n \text{ dense } n\ L^2(\mathbb{R})$$

then

$$\{D_{2^n}T_\ell\psi : n, \ell \in \mathbb{Z}\}$$

is an orthonormal basis of $L^2(\mathbb{R})$*.*

(If we have the assumptions on ϕ *and the* V_n *needed here, then we say that we have a* ***multiresolution analysis*** *of* $L^2(\mathbb{R})$*.)*

Proof.

(i) We have defined

$$\psi = \sum_{k\in\mathbb{Z}}(-1)^k\overline{c_{1-k}}D_2T_k\psi$$

Since T_ℓ is a continuous linear operator (in fact an isometry of $L^2(\mathbb{R})$), it follows that

$$\begin{aligned}
T_\ell\psi &= \sum_{k\in\mathbb{Z}}(-1)^k\overline{c_{1-k}}T_\ell D_2T_k\psi \\
&= \sum_{k\in\mathbb{Z}}(-1)^k\overline{c_{1-k}}D_2T_{2\ell}T_k\psi \\
&= \sum_{k\in\mathbb{Z}}(-1)^k\overline{c_{1-k}}D_2T_{k+2\ell}\psi \\
&\qquad \text{put } j = k+2\ell \\
&= \sum_{j\in\mathbb{Z}}(-1)^{j-2\ell}\overline{c_{1-j+2\ell}}D_2T_j\psi \\
&= \sum_{j\in\mathbb{Z}}(-1)^{j}\overline{c_{1-j+2\ell}}D_2T_j\psi
\end{aligned}$$

Now, expanding the inner product and using orthonormality of the $D_2T_k\phi$, we have

$$\langle\psi, T_\ell\psi\rangle = \sum_k(-1)^k\overline{c_{1-k}}\,\overline{(-1)^k\overline{c_{1-k+2\ell}}} = \sum_k\overline{c_{1-k}}c_{1-k+2\ell}.$$

If $\ell = 0$, then we get

$$\sum_k c_{1-k}\overline{c_{1-k}} = \sum_j |c_j|^2 = 1.$$

Thus $\langle\psi,\psi\rangle = 1$ or $\|\psi\| = 1$.

On the other hand, if $\ell \neq 0$, then

$$\langle\psi, T_\ell\psi\rangle = \sum_k\overline{c_{1-k}}c_{1-k+2\ell} = \sum_j\overline{c_{j-2\ell}}c_j = 0$$

(by (ii) of Proposition 1.5).

(ii) We have

$$V_0 = \overline{\text{span}}\{T_k\phi : k \in \mathbb{Z}\}$$

and $V_1 = D_2 V_0$. Since D_2 is an isometry of $L^2(\mathbb{R})$ and $\{T_k\phi : k \in \mathbb{Z}\}$ is an orthonormal basis of V_0, it follows that $\{D_2 T_k\phi : k \in \mathbb{Z}\}$ is an orthonormal basis of V_1.

Moreover we can say that

$$\begin{aligned} V_0 &= \left\{ \sum_{k\in\mathbb{Z}} a_k T_k\phi : \sum_k |a_k|^2 < \infty \right\} \\ V_1 &= \left\{ \sum_{k\in\mathbb{Z}} a_k D_2 T_k\phi : \sum_k |a_k|^2 < \infty \right\} \end{aligned}$$

Since $\phi = \sum_k c_k D_2 T_k \phi$ it follows that $\phi \in V_1$, and then

$$T_\ell\phi = \sum_{k\in\mathbb{Z}} c_k D_2 T_{k+2\ell}\phi = \sum_{k\in\mathbb{Z}} c_{k-2\ell} D_2 T_k\phi \Rightarrow T_\ell \in V_1 \forall \ell \in \mathbb{Z}.$$

It follows that $V_0 \subseteq V_1$.

$$\psi = \sum_{k\in\mathbb{Z}} (-1)^k \overline{c_{1-k}} D_2 T_k \psi \in V_1$$

is clear, and

$$\begin{aligned} T_\ell\psi &= \sum_{k\in\mathbb{Z}} (-1)^k \overline{c_{1-k}} T_\ell D_2 T_k \\ &= \sum_{k\in\mathbb{Z}} (-1)^k \overline{c_{1-k}} D_2 T_{k+2\ell} \\ &= \sum_{k\in\mathbb{Z}} (-1)^{k-2\ell} \overline{c_{1-k+2\ell}} D_2 T_k \\ &\in V_1 \end{aligned}$$

To show that $\{T_k\phi : k \in \mathbb{Z}\} \cup \{T_\ell\psi : \ell \in \mathbb{Z}\}$ is orthonormal, consider three types of inner products $\langle T_k\phi, T_j\phi\rangle$, $\langle T_\ell\psi, T_j\psi\rangle$ and $\langle T_k\phi, T_\ell\phi\rangle$. For the first two, we already know

$$\langle T_k\phi, T_j\phi\rangle = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

$$\langle T_\ell\psi, T_j\psi\rangle = \begin{cases} 1 & \text{if } \ell = j \\ 0 & \text{if } \ell \neq j \end{cases}$$

Next we show $\langle T_k\phi, T_\ell\phi\rangle = 0$ always, but first consider $\langle \phi, T_\ell\phi\rangle$. Working with $\phi = \sum_k c_k D_2 t_k \phi$ and $T_\ell\psi = \sum_k (-1)^k \overline{c_{1-k+2\ell}} D_2 T_k \phi$, and using the fact that $\{D_2 T_k \phi : k \in \mathbb{Z}\}$ is orthonormal, we have

$$\begin{aligned} \langle \phi, T_\ell \phi\rangle &= \sum_k c_k \overline{(-1)^k \overline{c_{1-k+2\ell}}} \\ &= \sum_k (-1)^k c_k c_{1-k+2\ell} \end{aligned}$$

In this sum consider the terms with $k = j$ and $k = 1 - j + 2\ell$ ($\Rightarrow 1 - k + 2\ell = j$)

$$\begin{aligned} &(-1)^j c_j c_{1-j+2\ell} + (-1)^{1-j+2\ell} c_{1-j+2\ell} c_j \\ &= (-1)^j c_j c_{1-j+2\ell} + (-1)^{1+j} c_{1-j+2\ell} c_j \\ &= 0. \end{aligned}$$

(In other words, for each term of the sum, there is another term which is (-1) times it.) Thus $\langle \phi, T_\ell \psi\rangle = 0$ for all $\ell \in \mathbb{Z}$.

Applying T_k to this fact (and using the fact that T_k is an isometry and therefore preserves inner products), we get

$$\begin{aligned} \langle T_k\phi, T_k T_\ell \psi\rangle &= 0 \\ \langle T_k\phi, T_{k+\ell} \psi\rangle &= 0 \quad \forall k, \ell \in \mathbb{Z} \\ \langle T_k\phi, T_\ell \psi\rangle &= 0 \quad \forall k, \ell \in \mathbb{Z} \end{aligned}$$

Now we know

$$\{T_k\phi : k \in \mathbb{Z}\} \cup \{T_\ell \psi : \ell \in \mathbb{Z}\}$$

is orthonormal in V_1. We still have to show it spans V_1.

Here is a matrix proof of that. (There are other proofs in books.)

Work with coefficients with respect to the basis $\{D_2 T_j \phi : j \in \mathbb{Z}\}$ of V_1.

Write $\Phi_k =$ the column vector of coefficients of $T_k\phi$

$$\Phi_k = [c_{j-2k}]_{j=+\infty}^{-\infty} = \begin{bmatrix} \vdots & \\ c_{-2-2k} & \\ c_{-1-2k} & \\ c_{0-2k} & \leftarrow 0 \text{ position} \\ c_{1-2k} & \\ c_{2-2k} & \\ \vdots & \end{bmatrix}$$

$\Psi_\ell =$ the coefficients of ψ_ℓ,

$$\Psi_\ell = [(-1)^j \overline{c_{1-j+2\ell}}]_{j=+\infty}^{-\infty} = \begin{bmatrix} \vdots & \\ (-1)^{-1}\overline{c_{2+2\ell}} & \\ (-1)^0\overline{c_{1+2\ell}} & \leftarrow 0 \\ (-1)^1\overline{c_{2\ell}} & \\ \vdots & \end{bmatrix}$$

Form an $\infty \times \infty$ matrix M (indexed by $\mathbb{Z}$ in both directions) with columns

$$(\ldots, \Phi_{-1}, \Psi_{-1}, \Phi_0, \Psi_0, \Phi_1, \Psi_1, \Phi_2, \Psi_2, \ldots)$$

(with Φ_0 in column 0).

Then M looks like

$$\begin{bmatrix} & \begin{array}{cccc} 0 & & & \\ \downarrow & & & \\ \Phi_0 & \Psi_0 & \Phi_1 & \Psi_1 \end{array} \\[2ex] & \begin{array}{cccc} M_{0,0} & & M_{0,1} & \end{array} \\ 0 \rightarrow & \begin{array}{|cc|cc|} \hline c_0 & \overline{c_1} & c_{-2} & \overline{c_{-1}} \\ c_1 & -\overline{c_0} & c_{-1} & \overline{c_{-2}} \\ \hline \end{array} \\ & \begin{array}{cccc} M_{1,0} & & M_{1,1} & \end{array} \\ & \begin{array}{|cc|cc|} \hline c_2 & \overline{c_{-1}} & c_0 & \overline{c_1} \\ c_3 & -\overline{c_{-2}} & c_1 & \overline{c_0} \\ \hline \end{array} \end{bmatrix}$$

If we work out M^*M we get the identity matrix (1's on the diagonal, 0's off it) since the rows of M^* are the complex conjugates of the columns of $M =$ the complex conjugates of the Φ_k's and the Ψ_ℓ's (written as rows) and when these are multiplied into the columns of M we get inner products between ϕ_k and ψ_ℓ with others of them.

We need to know $MM^* =$ identity, but this does *not* follow automatically from $M^*M =$ identity for infinite matrices.

M is made of 2×2 blocks with

$$M_{rs} = \begin{bmatrix} c_{2r-2s} & \overline{c_{2s-2r+1}} \\ c_{2r-2s+1} & -\overline{c_{2s-2r}} \end{bmatrix}$$

M^* will then be made of 2×2 blocks with the (r, s) block of M^* equal to $(M_{s,r})^*$.

If we multiply M^*M using blocks then the (r, s) block of the product will be

$$\begin{aligned}
&\sum_t ((r,t) \text{ block of } M)((t,s) \text{ block of } M^*) \\
&= \sum_t M_{r,t}(M_{s,t})^* \\
&= \sum_t \begin{bmatrix} c_{2r-2t} & \overline{c_{2t-2r+1}} \\ c_{2r-2t+1} & -\overline{c_{2t-2r}} \end{bmatrix} \begin{bmatrix} \overline{c_{2s-2t}} & \overline{c_{2s-2t+1}} \\ c_{2t-2s+1} & -c_{2t-2s} \end{bmatrix} \\
&= \sum_t \begin{bmatrix} c_{2r-2t}\overline{c_{2s-2t}} + \overline{c_{2t-2r+1}}c_{2t-2s+1} & c_{2r-2t}\overline{c_{2s-2t+1}} - \overline{c_{2t-2r+1}}c_{2t-2s} \\ c_{2r-2t+1}\overline{c_{2s-2t}} - \overline{c_{2t-2r}}c_{2t-2s+1} & c_{2r-2t+1}\overline{c_{2s-2t+1}} + \overline{c_{2t-2r}}c_{2t-2s} \end{bmatrix}
\end{aligned}$$

Bring the sums inside the matrix.

The off-diagonal sums rearrange to 0 and the orthogonality relations for the c's show that the diagonal entries are 0 unless $r = s$, when they are 1.

For example,

$$\begin{aligned}
&\sum_t c_{2r-2t}\overline{c_{2s-2t+1}} - \overline{c_{2t-2r+1}}c_{2t-2s} \\
&= \sum_t c_{2r-2t}\overline{c_{2s-2t+1}} - \sum_t \overline{c_{2t-2r+1}}c_{2t-2s} \\
&= \sum_u c_{2u}\overline{c_{2u+2(s-r)+1}} - \sum_v c_{2v}\overline{c_{2v+2(s-r)+1}} \\
&\quad \text{using } u = r - t \text{ in the first summation} \\
&\quad \text{and } v = t - s \text{ in the second} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
&\sum_t c_{2r-2t}\overline{c_{2s-2t}} + \overline{c_{2t-2r+1}}c_{2t-2s+1} \\
&= \sum_t c_{2r-2t}\overline{c_{2s-2t}} + \sum_t \overline{c_{2t-2r+1}}c_{2t-2s+1} \\
&= \sum_u c_{2u}\overline{c_{2(s-r)+2u}} + \sum_v c_{2v+1}\overline{c_{2v+2(s-r)+1}} \\
&= \sum_w c_w\overline{c_{w+2(s-r)}} \\
&= \langle \phi, T_{r-s}\phi \rangle
\end{aligned}$$

Hence MM^* is the identity.

Now suppose we have $f \in V_1$ perpendicular to each of the $T_k\phi$ and $T_\ell\psi$. If we write F for the column vector of coefficients of f in the basis $D_2T_j\phi$ of V_1, then $M^*F = 0$.

Hence $F = MM^*F = 0$ and so $f = 0$. Thus the collection $\{T_k\phi : k \in \mathbb{Z}\} \cup \{T_\ell\psi : \ell \in \mathbb{Z}\}$ is a maximal orthonormal subset (an orthonormal basis) of V_0.

(iii) Consider $V_n = D_{2^n}V_0$ (as introduced in part (iv) of the statement) for $n \in \mathbb{Z}$.

We know $V_0 = D_{2^0}V_0 = D_1V_0 \subseteq V_1 = D_2V_0$ (that is $V_0 \subseteq V_1$). Apply D_{2^n} to this and we get

$$D_{2^n}V_0 \subseteq D_{2^n}V_1 = D_{2^n}D_2V_0 = D_{2^{n+1}}V_0.$$

(For this we rely on the fact that $D_\lambda D_\mu = D_{\lambda\mu}$ holds if $\lambda, \mu > 0$, a fact that is relatively simple to check, as follows. Take $f \in L^2(\mathbb{R})$ and let $g = D_\mu f$ so that $g(x) = \sqrt{\mu}f(\mu x)$. Then

$$\begin{aligned}
(D_\lambda D_\mu f)(x) &= (D_\lambda g)(x) \\
&= \sqrt{\lambda}g(\lambda x) \\
&= \sqrt{\lambda}\sqrt{\mu}f(\mu\lambda x) \\
&= \sqrt{\lambda\mu}f((\lambda\mu)x) \\
&= (D_{\lambda\mu}f)(x). \quad)
\end{aligned}$$

Thus we conclude that $V_n \subseteq V_{n+1}$ for all n. Each $T_\ell\psi \in V_1$ and we know $\{T_\ell\psi : \ell \in \mathbb{Z}\}$ is orthonrmal. That is

$$\langle T_\ell\psi, T_j\psi\rangle = \begin{cases} 1 & \text{if } \ell = j \\ 0 & \text{if } \ell \neq j \end{cases}$$

As D_{2^n} is an isometry (preserves inner products), it follows that

$$\langle D_{2^n}T_\ell\psi, D_{2^n}T_j\psi\rangle = \langle T_\ell\psi, T_j\psi\rangle = \begin{cases} 1 & \text{if } \ell = j \\ 0 & \text{if } \ell \neq j \end{cases}$$

From (ii) we know that $\langle T_\ell\psi, T_k\phi\rangle = 0$ for all k, ℓ. But $V_0 =$ closure of the span of $\{T_k\phi : k \in \mathbb{Z}\}$ and so we can say that $T_\ell\psi \perp V_0$ for all $\ell \in \mathbb{Z}$.

Applying D_{2^n} we we that $D_{2^n}T_\ell\psi \perp D_{2^n}V_0 = V_n$. Also $D_{2^n}T_\ell\psi \in D_{2^n}V_1 = V_{n+1}$. Thus

$$\langle D_{2^m}T_\ell\psi, D_{2^n}T_\ell\psi\rangle = 0 \text{ if } n = m+1$$

because then $D_{2^m}T_\ell\psi \in V_{m+1} = V_n$. More generally, for $m < n$ we have $D_{2^m}T_\ell\psi \in V_{m+1} \subseteq V_n$ since $V_{m+1} \subseteq V_{m+2} \subseteq \cdots \subseteq V_n$ (note $m+1 \leq n$), and so we have

$$\langle D_{2^m}T_\ell\psi, D_{2^n}T_\ell\psi\rangle = 0 \text{ if } m < n.$$

This shows (iii).

(iv) We are now assuming that $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$ and that $\bigcup_{n\in\mathbb{Z}} V_n$ is dense in $L^2(\mathbb{R})$ and what we have to show is that $\text{span}\{D_{2^n}T_\ell\psi : n, \ell \in \mathbb{Z}\}$ is dense in $L^2(\mathbb{R})$.

(We know it is orthonormal, but to show that it is an orthonormal basis we have to show that it is maximal — that is, that no more functions in $L^2(\mathbb{R})$ can be added to it and keep the set orthonormal.)

If the span is not dense (or if the set $\{D_{2^n}T_\ell\psi : n, \ell \in \mathbb{Z}\}$ is not maximal) we can find $f \in L^2(\mathbb{R})$ with $f \neq 0$ (or even $\|f\| = 1$) so that

$$\langle f, D_{2^n}T_\ell\psi\rangle = 0 \quad \forall n, \ell.$$

Since $\bigcup_{n\in\mathbb{Z}} V_n$ is dense in $L^2(\mathbb{R})$ we cannot have $f \perp V_n$ for all n ($f \perp V_n \forall n \Rightarrow f \perp \bigcup_{n\in\mathbb{Z}} V_n \Rightarrow f \perp$ to the closure of $\bigcup_{n\in\mathbb{Z}} V_n = L^2(\mathbb{R}) \Rightarrow f \perp f \Rightarrow f = 0$). Fix an n with f not perpendicular to V_n.

Let P_n denote the orthogonal projection of $L^2(\mathbb{R})$ onto V_n. Since V_n has orthonormal basis $\{D_{2^n}T_j\phi : j \in \mathbb{Z}\}$ we can write

$$P_n f = \sum_{j\in\mathbb{Z}} \langle f, D_{2^n}T_j\phi\rangle D_{2^n}T_j\phi$$

and $P_n f \neq 0$.

For $m \geq n$, we have $D^{2^m}T_\ell\psi \perp V_m \supseteq V_n$ and so $P_n f \perp D^{2^m}T_\ell\psi \forall \ell \in \mathbb{Z}, \forall m \geq n$.

Put

$$\begin{aligned} W_n &= \text{the orthogonal complement of } V_{n-1} \text{ inside } V_n \\ &= \text{closure of the span of } \{D^{2^n}T_\ell\psi : \ell \in \mathbb{Z}\} \end{aligned}$$

(using (ii)).

Now $f - P_n f \in V_n^{\perp}$ and so $\langle f - P_n f, g\rangle = 0$ for $g \in V_n$. Or $\langle f, g\rangle = \langle P_n f, g\rangle \forall g \in V_n$. If $m < n$ then $D^{2^m} T_\ell \psi \in V_{m+1} \subseteq V_n$ and so

$$\langle P_n f, D^{2^m} T_\ell \psi\rangle = \langle f, D^{2^m} T_\ell \psi\rangle = 0 \forall m < n$$

$P_n f \in V_n = V_{n-1} \otimes W_n$ (orthogonal direct sum) and $P_n f \perp W_n$. Thus $P_n f \in V_{n-1} = V_{n-2} \otimes W_{n-1}$. By similar reasoning, $P_n f \in V_{n-2}$. Continuing by induction we get

$$\begin{aligned} P_n f \in \bigcap_{j=1}^{\infty} V_{n-j} &= \bigcap_{m=-\infty}^{n} V_m \\ &= \bigcap_{m=-\infty}^{\infty} V_m \\ &\quad \text{since } V_m \subseteq V_{m+1} \forall m \\ &= \{0\} \end{aligned}$$

But this is a contradiction since $P_n f \neq 0$.

We conclude that it is impossible to find a nonzero $f \perp D_{2^n} T_\ell \psi \forall n, \ell \in \mathbb{Z}$.

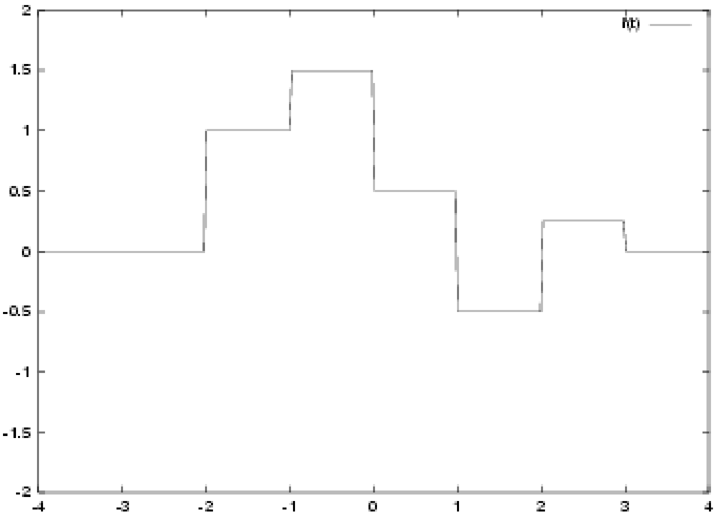
Therefore $\{D_{2^n} T_\ell \psi : n, \ell \in \mathbb{Z}\}$ spans a dense subspace of $L^2(\mathbb{R})$ and is an orthonormal basis. □

Example 1.12 (Haar case) *In the Haar case we do have a multiresolution analysis.*

$$\phi = \chi_{[0,1)}$$

$$\begin{aligned} V_0 &= \text{closed span of } \{T_k \phi : k \in \mathbb{Z}\} \\ &= \text{closed span of } \{\chi_{[k,k+1)} : k \in \mathbb{Z}\} \\ &= \text{functions constant on the intervals } [k, k+1) \\ &\quad \text{and in } L^2(\mathbb{R}) \end{aligned}$$

Finite linear combinations $\sum_{k=k_0}^{k_1} a_k \chi_{[k,k+1)}$ *are the step functions with steps at the integers only (and compact support) and functions in the* $L^2(\mathbb{R})$ *closure will be almost everywhere constant on each* $[k, k+1)$*.*



$$
\begin{aligned}
V_1 &= D_2 V_0 \\
&= \{\sqrt{2} f(2x) : f \in V_0\} \\
&= \textit{functions constant almost everywhere on each} \\
&\qquad \textit{of the intervals } \left[\frac{k}{2}, \frac{k+1}{2}\right) \\
V_n &= D_{2^n} V_0 \\
&= \textit{functions constant almost everywhere on each} \\
&\qquad \textit{of the intervals } \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)
\end{aligned}
$$

If $n \to \infty$ *the intervals get shorter and* $\bigcap_{n\in\mathbb{Z}} V_n$ *is dense in* $L^2(\mathbb{R})$*. If* $n \to -infty$ *the intervals get longer and no nonzero* $L^2(\mathbb{R})$ *function can be in* $\bigcap_{n\in\mathbb{Z}} V_n$ *because such a function would e constant on* $[0, 2^n) = \left(\left[\frac{k}{2^{-n}}, \frac{k+1}{2^{-n}}\right)\right.$ *with* $k = 0$*) and on* $[-2^n, 0)$ *for all* n*. Thus it would have to be constant on* $[0, \infty)$ *and on* $(-\infty, 0)$*. Thus it would be* $\alpha\chi_{[0,\infty)} + \beta\chi_{(-\infty,0)}$ *which cannot be in* $L^2(\mathbb{R})$ *unless* $\alpha = \beta = 0$*.*

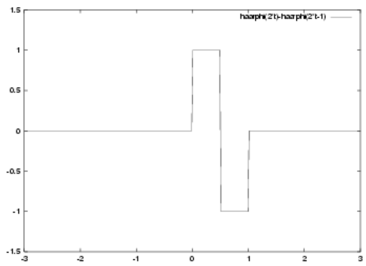
All the properties for a multiresolution analysis are satisfied. Recall that $\phi = \chi_{[0,1)}$ *has orthonormal translates and satisfies a dilation equation with two nonzero coefficients* $c_0 = c_1 = 1/\sqrt{2}$

$$\phi(x) = c_0\sqrt{2}\phi(2x) + c_1\sqrt{2}\phi(2x-1).$$

The wavelet construction gives

$$\psi = \sum_{k\in\mathbb{Z}} (-1)^k \overline{c_{1-k}} D_2 T_k \phi$$

$$
\begin{aligned}
&= (-1)^0\overline{c_1}D_2T_0\phi + (-1)^1\overline{c_0}D_2T_1\phi \\
\psi(x) &= (1/\sqrt{2})\sqrt{2}\phi(2x) - (1/\sqrt{2})\sqrt{2}\phi(2x-1) \\
&= \phi(2x) - \phi(2x-1) \\
&= \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x)
\end{aligned}
$$



The theorem above (1.11) says that we can get other wavelets, but we need to be able to check the hypotheses on the V_n's. For the Daubechies 4 coefficient case

$$c_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}, c_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}, c_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}, c_3 = \frac{1-\sqrt{3}}{4\sqrt{2}},$$

it is true that all these hypotheses work out for a compactly supported continuous ϕ, but we have not yet proved that. The proof will go back to the infinite product formula for $\mathcal{F}(\phi)$.

With finitely many nonzero c_k and $p(\xi)$ the trigonometric polynomial

$$p(\xi) = \sum_{k\in\mathbb{Z}} c_k e^{-2\pi i k\xi},$$

we have

$$\mathcal{F}\phi(\xi) = \mathcal{F}\phi(0) \lim_{n\to\infty} \prod_{j=1}^{n} \frac{1}{\sqrt{2}} p\left(\frac{\xi}{2^j}\right)$$

and $\mathcal{F}\phi(0) = 1$ will be true in many cases.

Here is a fact about p that we will use later.

Lemma 1.13 *Assuming the c_k satisfy the orthogonality conditions*

$$\sum_{k\in\mathbb{Z}} c_k\overline{c_{k-2\ell}} = \begin{cases} 1 & \text{for } \ell = 0 \\ 0 & \text{for } \ell \in \mathbb{Z}, \ell \neq 0. \end{cases}$$

(and only a finite number of k with c_k nonzero) then

$$|p(\xi)|^2 + |p(\xi+1/2)|^2 = 2$$

Proof. One can prove this based on the existence of a solution ϕ to the dilation equation and showing that the Fourier transform of ϕ must satisfy

$$\sum_{\ell=-\infty}^{\infty} |\mathcal{F}\phi(\xi+\ell)|^1 \equiv 1,$$

but a much more direct proof is possible.

From the definition of $p(\xi)$ we compute first

$$\begin{aligned}
|p(\xi)|^2 &= p(\xi)\overline{p(\xi)} \\
&= \left(\sum_{k\in\mathbb{Z}} c_k e^{-2\pi i k\xi}\right)\left(\sum_{\ell\in\mathbb{Z}} \overline{c_\ell} e^{2\pi i \ell\xi}\right) \\
&\quad \text{(only finitely many nonzero terms} \\
&\quad \text{in each sum)} \\
&= \sum_{k,\ell} c_k\overline{c_\ell} e^{-2\pi i(k-\ell)\xi} \\
&= \sum_m \sum_{k,\ell \text{ with } k-\ell=m} c_k\overline{c_\ell} e^{-2\pi i m\xi} \\
&= \sum_m \left(\sum_k c_k \overline{c_{k-m}}\right) e^{-2\pi i m\xi} \\
|p(\xi)|^2 + |p(\xi+1/2)|^2 &= \sum_m \left(\sum_k c_k \overline{c_{k-m}}\right) e^{-2\pi i m\xi} \\
&\quad + \sum_m \left(\sum_k c_k \overline{c_{k-m}}\right) e^{-2\pi i m(\xi+1/2)}
\end{aligned}$$

$$\begin{aligned}
e^{-2\pi i m(\xi+1/2)} &= e^{-2\pi i m\xi} e^{-\pi i m} \\
&= (e^{-\pi i})^m e^{-2\pi i m\xi} \\
&= (-1)^m e^{-2\pi i m\xi} \\
e^{-2\pi i m\xi} + e^{-2\pi i m(\xi+1/2)} &= \begin{cases} 2e^{-2\pi i m\xi} & \text{if } m \text{ even} \\ 0 & \text{if } m \text{ odd} \end{cases}
\end{aligned}$$

$$|p(\xi)|^2 + |p(\xi+1/2)|^2 = 2\sum_{m \text{ even}} \left(\sum_k c_k \overline{c_{k-m}}\right) e^{-2\pi i m\xi}$$

$$\begin{aligned}
&= 2\sum_{\ell}\left(\sum_k c_k\overline{c_{k-2\ell}}\right)e^{-2\pi i(2\ell)\xi} \\
&= 2e^0 \text{ by the orthogonality relations} \\
&= 2
\end{aligned}$$

Remark 1.14 *We can express $\mathcal{F}\psi(\xi)$ in terms of $\mathcal{F}\phi(\xi/2)$ and p.*

$$\begin{aligned}
\psi &= \sum_{k\in\mathbb{Z}}(-1)^k\overline{c_{1-k}}D_2T_k\phi \\
\mathcal{F}\psi &= \sum_{k\in\mathbb{Z}}(-1)^k\overline{c_{1-k}}\mathcal{F}(D_2T_k\phi)
\end{aligned}$$

Now we can use

$$\begin{aligned}
\mathcal{F}(D_2f) &= D_{1/2}(\mathcal{F}f) \\
\mathcal{F}(D_2f)(\xi) &= \sqrt{\frac{1}{2}}(\mathcal{F}f)\left(\frac{\xi}{2}\right) \\
\mathcal{F}(T_kf)(\xi) &= e^{-2\pi ik\xi}(\mathcal{F}f)(\xi) \\
\mathcal{F}(D_2T_kf) &= \sqrt{\frac{1}{2}}e^{-2\pi ik(\xi/2)}(\mathcal{F}f)(\xi/2)
\end{aligned}$$

and get

$$\begin{aligned}
(\mathcal{F}\psi)(\xi) &= \sum_{k\in\mathbb{Z}}(-1)^k\overline{c_{1-k}}\sqrt{\frac{1}{2}}e^{-2\pi ik(\xi/2)}(\mathcal{F}\phi)(\xi/2) \\
&= \left(\sum_{k\in\mathbb{Z}}\left(e^{-i\pi}\right)^k\overline{c_{1-k}}\sqrt{\frac{1}{2}}e^{-2\pi ik(\xi/2)}\right)(\mathcal{F}\phi)(\xi/2) \\
&= \sqrt{\frac{1}{2}}\left(\sum_{k\in\mathbb{Z}}\overline{c_{1-k}}e^{-2\pi ik\left(\frac{\xi+1}{2}\right)}\right)(\mathcal{F}\phi)(\xi/2)
\end{aligned}$$

Note that

$$\begin{aligned}
p(\xi) &= \sum_{k\in\mathbb{Z}}c_ke^{-2\pi ik\xi} \\
\overline{p(\xi)} &= \sum_{k\in\mathbb{Z}}\overline{c_k}e^{2\pi ik\xi} \\
&= \sum_{k\in\mathbb{Z}}\overline{c_{1-k}}e^{2\pi i(1-k)\xi}
\end{aligned}$$

$$\begin{aligned} &= e^{2\pi i\xi}\sum_{k\in\mathbb{Z}}\overline{c_{1-k}}e^{-2\pi ik\xi} \\ e^{-2\pi i\xi}\overline{p(\xi)} &= \sum_{k\in\mathbb{Z}}\overline{c_{1-k}}e^{-2\pi ik\xi} \end{aligned}$$

and thus

$$(\mathcal{F}\psi)(\xi) = \sqrt{\frac{1}{2}}e^{-2\pi i\left(\frac{\xi+1}{2}\right)}\overline{p\left(\frac{\xi+1}{2}\right)}(\mathcal{F}\phi)(\xi/2)$$

Recall that we also have the Fourier transform of the dilation equation as

$$(\mathcal{F}\phi)(\xi) = \sqrt{\frac{1}{2}}p\left(\frac{\xi}{2}\right)(\mathcal{F}\phi)(\xi/2)$$

The two multipliers on the right of these equations

$$\sqrt{\frac{1}{2}}e^{-2\pi i\left(\frac{\xi+1}{2}\right)}\overline{p\left(\frac{\xi+1}{2}\right)} \text{ and } \sqrt{\frac{1}{2}}p\left(\frac{\xi}{2}\right)$$

have the sum of the squares of their absolute values equal to

$$\frac{1}{2}\left|p\left(\frac{\xi+1}{2}\right)\right|^2 + \frac{1}{2}\left|p\left(\frac{\xi}{2}\right)\right|^2 = 1$$

by Lemma 1.13.

This fact can be used as the basis for a different explanation for the orthogonal decomposition

$$V_1 = V_0 \oplus W_1$$

by looking at things from the point of view of the Fourier transform.

Lemma 1.15 *If ϕ satisfies a two scale dilation equation*

$$\phi = \sum_k c_k D_2 T_k \phi,$$

has orthonormal translates $T_k\phi$ ($k \in \mathbb{Z}$), gives rise to a multiresolution analysis by $V_0 =$ the closure of the span of $\{T_k\phi : k \in \mathbb{Z}\}$, $V_n = D_{2^n}V_0$ ($n \in \mathbb{Z}$) and if $\phi \in L^1(\mathbb{R} \cap L^2(\mathbb{R})$, then

$$\left|\int_{-\infty}^{\infty}\phi(x)\,dx\right| = 1$$

Proof. Let P_n denote the orthogonal projection of $L^2(\mathbb{R})$ onto V_n, so that

$$P_n f = \sum_{k\in\mathbb{Z}} \langle f, D_{2^n} T_k \phi\rangle D_{2^n} T_k \phi.$$

Choose $f \in L^2(\mathbb{R})$ so that

$$\mathcal{F} f = \chi_{[-1,1]}$$

(This requires the extension of $\mathcal{F}$ to an isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ — see Appendix A.)

Now compute

$$\begin{aligned}
\|P_n f\|_2^2 &= \sum_{k\in\mathbb{Z}} |\langle f, D_{2^n} T_k \phi\rangle|^2 \\
&= \sum_{k\in\mathbb{Z}} |\langle \mathcal{F} f, \mathcal{F}(D_{2^n} T_k \phi)\rangle|^2 \\
&= \sum_{k\in\mathbb{Z}} \left| \int_{-1}^{1} \overline{\mathcal{F}(D_{2^n} T_k \phi)(\xi)} \right|^2 \\
&= \sum_{k\in\mathbb{Z}} \left| \overline{\int_{-1}^{1} \mathcal{F}(D_{2^n} T_k \phi)(\xi)} \right|^2 \\
&= \sum_{k\in\mathbb{Z}} \left| \int_{-1}^{1} \mathcal{F}(D_{2^n} T_k \phi)(\xi) \right|^2
\end{aligned}$$

Recall

$$\begin{aligned}
\mathcal{F}(D_\lambda f) &= D_{1/\lambda}(\mathcal{F} f) \\
(\mathcal{F}(T_k f))(\xi) &= e^{-2\pi i k \xi} (\mathcal{F} f)(\xi) \\
\mathcal{F}(D_{2^n} T_k \phi)(\xi) &= D_{1/2^n} \left(\xi \mapsto (\mathcal{F}(T_k f))(\xi)\right) \\
&= \sqrt{\frac{1}{2^n}} e^{-2\pi i k (\xi/2^n)} (\mathcal{F}\phi)(\xi/2^n)
\end{aligned}$$

and so

$$\|P_n f\|_2^2 = \sum_{k\in\mathbb{Z}} \left| \int_{-1}^{1} \sqrt{\frac{1}{2^n}} e^{-2\pi i k (\xi/2^n)} (\mathcal{F}\phi)\left(\frac{\xi}{2^n}\right) d\xi \right|^2$$

$$\text{put } \eta = \frac{\xi}{2^n}$$

$$d\eta = \frac{d\xi}{2^n}$$

$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}} \left| \int_{-1/2^n}^{1/2^n} \sqrt{\frac{1}{2^n}} e^{-2\pi i k\eta} (\mathcal{F}\phi)(\eta) 2^n \, d\eta \right|^2 \\
&= 2^n \sum_{k\in\mathbb{Z}} \left| \int_{-1/2^n}^{1/2^n} e^{-2\pi i k\eta} (\mathcal{F}\phi)(\eta) \, d\eta \right|^2 \\
&= 2^n \sum_{k\in\mathbb{Z}} \left| \int_{-1/2}^{1/2} \chi_{[-1/2^n,1/2^n]}(\eta) (\mathcal{F}\phi)(\eta) e^{-2\pi i k\eta} \, d\eta \right|^2 \\
&\qquad (\text{if } n \geq 1)
\end{aligned}
$$

We know that $\left(e^{-2\pi i k\eta}\right)_{k\in\mathbb{Z}}$ is an orthonormal basis (used for Fourier series) of $L^2([0,1])$, but it also works for $L^2([-1/2,1/2])$. (This can be checked by repeating the proof for $L^2([0,1])$, or by noting that the change of variables $\eta \mapsto \eta - 1/2$ maps $L^2([0,1]) \to L^2([-1/2,1/2])$ isometrically and sends $e^{-2\pi i k\eta}$ to $(-1)^k e^{-2\pi i k\eta}$.)

This fact means that the last summation above is the sum of the squares of the absolute values of absolute values of the 'Fourier' coefficients of the function

$$\eta \mapsto \chi_{[-1/2^2,1/2^n]}(\eta)(\mathcal{F}\phi)(\eta)$$

in $L^2([-1/2,1/2])$.

Thus the summation is the square of the $L^2([-1/2,1/2])$ norm of the function and we have (assuming $n \geq 1$)

$$
\begin{aligned}
\|P_n f\|_2^2 &= 2^n \int_{-1/2}^{1/2} \left|\chi_{[-1/2^2,1/2^n]}(\eta)(\mathcal{F}\phi)(\eta)\right|^2 \, d\eta \\
&= 2^n \int_{-1/2^n}^{1/2^n} |(\mathcal{F}\phi)(\eta)|^2 \, d\eta \\
&\qquad \text{put } \xi = 2^n \eta \\
&\qquad\qquad d\xi = 2^n d\eta \\
&= \int_{-1}^{1} \left| (\mathcal{F}\phi)\left(\frac{\xi}{2^n}\right) \right|^2 \, d\xi
\end{aligned}
$$

Now $\phi \in L^1(\mathbb{R}) \Rightarrow \mathcal{F}\phi$ is continuous (at 0). Thus

$$\lim_{n\to\infty} (\mathcal{F}\phi)\left(\frac{\xi}{2^n}\right) = (\mathcal{F}\phi)(0)$$

uniformly for $\xi \in [-1,1]$ and so we conclude that

$$\lim_{n\to\infty} \|P_n f\|_2^2 = \int_{-1}^{1} |(\mathcal{F}\phi)(0)|^2 \, d\xi = 2|(\mathcal{F}\phi)(0)|^2$$

But $\bigcup_{n\in\mathbb{Z}} V_n$ dense in $L^2(\mathbb{R})$ (and $V_n \subseteq V_{n+1} \forall n$) allows us to conclude that

$$\lim_{n\to\infty} \|P_n f\|_2^2 = \|f\|_2^2 = \|\mathcal{F}f\|_2^2 = \left\|\chi_{[-1,1]}\right\|_2^2 = \int_{-1}^{1} 1\, d\xi = 2$$

(The reason is that $P_n f =$ the element of V_n closest to f and so $V_n \subseteq V_{n+1} \Rightarrow \|f - P_n f\|_2 \geq \|f - P_{n+1} f\|_2 \forall n$. As $\bigcup_{n\in\mathbb{Z}} V_n$ is dense in $L^2(\mathbb{R})$, we can find $g \in \bigcup_{n\in\mathbb{Z}} V_n$ with $\|f - g\|_2$ arbitrarily small. As $g \in V_n$ for some n, it follows that we can find n with $\|f - P_n f\|_2$ arbitrarily small. For all $m > n$, $\|f - P_m f\|_2$ will be no larger than $\|f - P_n f\|_2$ and so we can show that $\lim_{m\to\infty} \|f - P_m f\|_2 = 0$. Thus $\lim_{n\to\infty} \|P_n f\|_2 = \|f\|$.)

Now we conclude that

$$2|(\mathcal{F}\phi)(0)|^2 = 2 \Rightarrow |(\mathcal{F}\phi)(0)| = 1,$$

which means

$$\left|\int_{-\infty}^{\infty} \phi(x)\, dx\right| = 1.$$

Remark 1.16 *Note that the above Lemma 1.13 shows that the normalisations*

$$\|\phi\|_2 = 1 \text{ and } \int_{-\infty}^{\infty} \phi(x)\, dx = 1$$

are compatible if ϕ is a scaling function for a multiresolution analysis and if $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

This means that the graph we drew in 1.8 of the Daubechies 4 coefficient ϕ is normalised correctly (IF we can show that in that case there is a multiresolution analysis and that ϕ is compactly supported and continuous).

Proposition 1.17 *Suppose ϕ has orthonormal translates $\{T_k\phi : k \in \mathbb{Z}\}$, $V_0 = \overline{span}\{T_k\phi : k \in \mathbb{Z}\}$, $V_n = D_{2^n} V_0$ $(n \in \mathbb{Z})$ and suppose $\mathcal{F}\phi(\xi)$ is continuous at $\xi = 0$ and has $\mathcal{F}\phi \neq 0$.*

Then the linear span of $\bigcup_{n\in\mathbb{Z}} V_n$ is dense in $L^2(\mathbb{R})$.

(We do not need the dilation equation here.)

Proof. Assume the span is not dense and so there exists $f \in L^2(\mathbb{R})$, $f \neq 0$ with $f \perp \bigcup_{n\in\mathbb{Z}} V_n$. (If the span is not dense, then its closure is a proper closed subspace and so has a nonzero orthogonal complement.)

Choose $R > 0$ large and define $g \in L^2(\mathbb{R})$ by

$$\begin{aligned} \mathcal{F}g &= \left(\chi_{[-R,R]}\right) \mathcal{F}f \\ (\mathcal{F}g)(\xi) &= \chi_{[-R,R]}(\xi)(\mathcal{F}f)(\xi) \end{aligned}$$

(Note that $f \in L^2(\mathbb{R}) \Rightarrow \mathcal{F}f \in L^2(\mathbb{R}) \Rightarrow \chi_{[-R,R]}\mathcal{F}f \in L^2(\mathbb{R}) \Rightarrow g = \mathcal{F}^{-1}\left(\chi_{[-R,R]}\mathcal{F}f\right) \in L^2(\mathbb{R})$.) We can find such a g for any $R > 0$ and also

$$\begin{aligned} \|f-g\|_2 &= \|\mathcal{F}(f-g)\|_2 \\ &= \|\mathcal{F}f - \mathcal{F}g\|_2 \\ &< \varepsilon \end{aligned}$$

if R is large enough (and for any given $\varepsilon > 0$).

(In other words, we can say that g approximates f quite well and also g is what is known as *band limited* — $\mathcal{F}g$ is compactly supported in $|\xi| \leq R$.)

Consider again the orthogonal projection $P_n \colon L^2(\mathbb{R}) \to V_n$ and the formula

$$P_n h = \sum_{k\in\mathbb{Z}} \langle h, D_{2^n}T_k\phi\rangle D_{2^n}T_k\phi.$$

Since $f \perp V_n$, $P_n f = 0$ and then we have

$$\begin{aligned} P_n g &= P_n f + P_n(g-f) \\ \|P_n g\|_2 &\leq \|P_n f\|_2 + \|P_n(g-f)\|_2 \\ &= 0 + \|P_n(g-f)\|_2 \\ &\leq \|g-f\|_2 < \varepsilon. \end{aligned}$$

Now we compute (in a similar way as we did in the previous proof)

$$\begin{aligned} \|P_n g\|_2^2 &= \sum_{k\in\mathbb{Z}} |\langle g, D_{2^n}T_k\phi\rangle|^2 \\ &= \sum_{k\in\mathbb{Z}} |\langle \mathcal{F}g, \mathcal{F}(D_{2^n}T_k\phi)\rangle|^2 \\ &= \sum_{k\in\mathbb{Z}} \left| \int_{-R}^{R} (\mathcal{F}g)(\xi)\overline{\mathcal{F}(D_{2^n}T_k\phi)(\xi)} \right|^2 \\ &= \sum_{k\in\mathbb{Z}} \left| \int_{-R}^{R} (\mathcal{F}g)(\xi)\overline{\sqrt{\frac{1}{2^n}} e^{-2\pi i k(\xi/2^n)}(\mathcal{F}\phi)(\xi/2^n))}\, d\xi \right|^2 \\ &= \sum_{k\in\mathbb{Z}} \left| \int_{-R}^{R} (\mathcal{F}g)(\xi)\sqrt{\frac{1}{2^n}} e^{2\pi i k(\xi/2^n)}\overline{(\mathcal{F}\phi)(\xi/2^n)}\, d\xi \right|^2 \\ & \text{put } \eta = \frac{\xi}{2^n} \\ & \quad d\eta = \frac{d\xi}{2^n} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}} \left| \int_{-R/2^n}^{R/2^n} (\mathcal{F}g)(2^n\eta)\sqrt{\frac{1}{2^n}}e^{2\pi ik\eta}\overline{(\mathcal{F}\phi)(\eta)}2^n\,d\eta \right|^2 \\
&= \sum_{k\in\mathbb{Z}} \left| \int_{-1/2}^{1/2} (\mathcal{F}g)(2^n\eta)\overline{(\mathcal{F}\phi)(\eta)}\sqrt{2^n}e^{2\pi ik\eta}\,d\eta \right|^2 \\
&\qquad \text{if } 2^n > 2R \\
&\qquad \text{using } (\mathcal{F}g)(2^n\eta) = 0 \text{ if } |\eta| > R/2^n \\
&= \text{sum of the squares of the } (-k)\text{th coefficient of} \\
&\quad (\mathcal{F}g)(2^n\eta)\overline{(\mathcal{F}\phi)(\eta)}\sqrt{2^n} \\
&\quad \text{with respect to the othonormal basis} \\
&\quad \left(e^{2\pi ik\eta}\right)_{k=-\infty}^{\infty} \\
&\quad \text{of } L^2(\mathbb{R}) \\
&= \text{the } (L^2 \text{ norm})^2 \text{ of the function} \\
&= \int_{-1/2}^{1/2} |(\mathcal{F}g)(2^n\eta)|^2\,|(\mathcal{F}\phi)(\eta)|^2 2^n\,d\eta \\
&\quad \text{put } \xi = 2^n\eta \\
&\qquad d\xi = 2^n d\eta \\
&= \int_{-2^n/2}^{2^n/2} |(\mathcal{F}g)(\xi)|^2 \left|(\mathcal{F}\phi)\left(\frac{\xi}{2^n}\right)\right|^2 d\xi \\
&= \int_{-R}^{R} |(\mathcal{F}g)(\xi)|^2 \left|(\mathcal{F}\phi)\left(\frac{\xi}{2^n}\right)\right|^2 d\xi \\
&\quad (\text{again if } 2^n > 2R)
\end{aligned}
$$

Now $\mathcal{F}\phi$ is continuous at 0 and so $(\mathcal{F}\phi)\left(\frac{\xi}{2^n}\right) \to (\mathcal{F}\phi)(0)$ as $n \to \infty$ uniformly for $\xi \in [-R, R]$. Thus we conclude

$$
\begin{aligned}
\|P_n g\|_2^2 &\to \int_{-R}^{R} |(\mathcal{F}g)(\xi)|^2\,|(\mathcal{F}\phi)(0)|^2\,d\xi \\
&\qquad \text{as } n \to \infty \\
&= |(\mathcal{F}\phi)(0)|^2\|(\mathcal{F}g)\|_2^2 \\
&= |(\mathcal{F}\phi)(0)|^2\|g\|_2^2
\end{aligned}
$$

But $\|P_n g\|_2^2 < \varepsilon^2$ and so it follows that

$$
\begin{aligned}
|(\mathcal{F}\phi)(0)|^2\|g\|_2^2 &< \varepsilon^2 \\
\|g\|_2 &< \frac{\varepsilon}{|(\mathcal{F}\phi)(0)|} \\
\|f\|_2 &\le \|f-g\|_2 + \|g\|_2
\end{aligned}
$$

$$< \quad \varepsilon + \frac{\varepsilon}{|(\mathcal{F}\phi)(0)|}$$

As we can do this for $\varepsilon > 0$ arbitrarily small, it follows that $\|f\|_2 = 0$.

But this contradicts the choice of $f \neq 0$ and completes the proof (that the span must be dense). □

Proposition 1.18 *Assume ϕ has orthonormal translates $T_k\phi$ (in $L^2(\mathbb{R})$) and*

$$\begin{aligned} V_0 &= \overline{span}\{T_k\phi : k \in \mathbb{Z}\} \\ V_n &= D_{2^n} V_0 \end{aligned}$$

Then

$$\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$$

Proof. Let $P_n : L^2(\mathbb{R}) \to V_n$ be the orthogonal projection, so that

$$P_n f = \sum_{k\in\mathbb{Z}} \langle f, D_{2^n} T_k \phi \rangle D_{2^n} T_k \phi.$$

We will prove that

$$\lim_{n\to-\infty} P_n f = 0 \forall f \in L^2(\mathbb{R}) \tag{1}$$

and this implies the result because

$$\begin{aligned} f \in \bigcap_{n\in\mathbb{Z}} V_n &\Rightarrow P_n f = f \forall n \in \mathbb{Z} \\ &\Rightarrow 0 = \lim_{n\to-\infty} P_n f = f \\ &\Rightarrow f = 0 \end{aligned}$$

To prove (1) we prove it holds for compactly supported $f \in L^2(\mathbb{R})$. Then the general case follows because $f \in L^2(\mathbb{R}) \Rightarrow \chi_{[-N,N]} f \in L^2(\mathbb{R}) \forall N > 0$ and $\chi_{[-N,N]}(x) f(x) \neq 0$ only for $x \in [-N, N]$ so that $\chi_{[-N,N]} f$ is supported in $[-N, N]$ (and is in $L^2(\mathbb{R})$).

If we know (1) for the compactly supported case then we know

$$\lim_{n\to\infty} P_n \left(\chi_{[-N,N]} f\right) = 0.$$

But we can choose N so that

$$\left\| f - \chi_{[-N,N]} f \right\|_2 < \varepsilon$$

(for any pre-assigned $\varepsilon > 0$)[1]. Then

$$\begin{aligned}\|P_n f\|_2 &= \|P_n f - P_n\left(\chi_{[-N,N]} f\right) + P_n\left(\chi_{[-N,N]} f\right)\|_2 \\ &\leq \|P_n\left(f - \chi_{[-N,N]} f\right)\|_2 + \|P_n\left(\chi_{[-N,N]} f\right)\|_2 \\ &\leq \|f - \chi_{[-N,N]} f\|_2 + \|P_n\left(\chi_{[-N,N]} f\right)\|_2\end{aligned}$$

Now $\|f - \chi_{[-N,N]} f\|_2 < \varepsilon$ if N is chosen to be large enough and then by the compactly supported case of (1) $\|P_n\left(\chi_{[-N,N]} f\right)\|_2 < \varepsilon$ if n is small enough. Thus

$$\|P_n f\|_2 < 2\varepsilon$$

if n is small enough. This shows $\lim_{n\to-\infty} \|P_n f\|_2 = 0$.

Now, take $f \in L^2(\mathbb{R})$ with compact support in $[-R, R]$ (for some $R > 0$) and look at

$$\begin{aligned}\|P_n f\|_2 &= \sum_{k\in\mathbb{Z}} |\langle f, D_{2^n} T_k \phi\rangle|^2 \\ &= \sum_{k=-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(x) \overline{\sqrt{2^n}\phi(2^n x - k)}\, dx \right|^2 \\ &= \sum_{k=-\infty}^{\infty} \left| \int_{-R}^{R} f(x) \sqrt{2^n \phi(2^n x - k)}\, dx \right|^2 \\ &\qquad \text{use Cauchy-Schwarz inequality} \\ &\leq \sum_{k=-\infty}^{\infty} \left(\int_{-R}^{R} |f(x)|^2\, dx \right) \left(\int_{-R}^{R} 2^n |\phi(2^n x - k)|^2\, dx \right) \\ &\qquad \text{put } y = 2^n x - k \text{ in the last integral} \\ &\qquad dy = 2^n\, dx \\ &= \|f\|_2^2 \sum_{k=-\infty}^{\infty} \int_{-k-2^n R}^{-k+2^n R} |\phi(y)|^2\, dy\end{aligned}$$

Choose n small enough (recall $n \to -\infty$) so that $2^n R < 1/2$.

---

[1]

$$\begin{aligned}\|f - \chi_{[-N,N]} f\|_2^2 &= \int_{-\infty}^{\infty} \left|f(x) - \chi_{[-N,N]}(x) f(x)\right|^2\, dx \\ &= \int_{-\infty}^{\infty} \left(\chi_{(-\infty,-N)}(x) + \chi_{(N,\infty)}(x)\right) |f(x)|^2\, dx \\ &\to 0 \text{ as } N \to \infty\end{aligned}$$

by the Lebesgue dominated convergence theorem because the integrand tends to zero pointwise as $N \to \infty$ and is at most equal to the integrable $|f(x)|^2$.

Then the above integrals are over disjoint intervals

$$[-k - 2^n R, -K + 2^n R] \subseteq \left(-k - \frac{1}{2}, -k + \frac{1}{2}\right)$$

Put

$$U_n = \bigcup_{k=-\infty}^{\infty} [-k - 2^n R, -K + 2^n R]$$

and then we have

$$\begin{aligned} \|P_n f\|_2^2 &\leq \|f\|_2^2 \int_{U_n} |\phi(y)|^2 \, dy \\ &\qquad \text{(as long as } 2^n R < \frac{1}{2}) \\ &= \|f\|_2^2 \int_{-\infty}^{\infty} \chi_{U_n}(y) |\phi(y)|^2 \, dy \end{aligned}$$

As $n \to -\infty$, $\chi_{U_n}(y) \to 0$ for all y except $y \in \mathbb{Z}$. Thus

$$\chi_{U_n}(y) \to 0 \text{ almost everywhere on } \mathbb{R}$$

(as $\mathbb{Z}$ is countable and so has measure zero). From the Lebesgue dominated convergence theorem we can conclude that

$$\lim_{n \to -\infty} \int_{-\infty}^{\infty} \chi_{U_n}(y) |\phi(y)|^2 \, dy = 0$$

because the integrands are $\leq |\phi(y)|^2$ for all y, $\int_{-\infty}^{\infty} |\phi(y)|^2 \, dy < \infty$ and the integrands $\to 0$ pointwise almost everywhere.

It follows that

$$\lim_{n \to -\infty} \|P_n f\|_2^2 = 0$$

Note. We still need to show that certain dilation equations have compactly supported solutions in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and even continuous compactly supported solutions.

Proposition 1.19 *Assume $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ satisfies a finite dilation equation*

$$\phi = \sum_{k=k_0}^{k_1} c_k D_2 T_k \phi,$$

that ϕ has orthonormal translates $\{T_k \phi : k \in \mathbb{Z}\}$ and $\int_{-\infty}^{\infty} \phi(x) \, dx \neq 0$. Then

$$\sum_{k \text{ even}} c_k = \sum_{k \text{ odd}} c_k.$$

Proof. We know from 1.5 (i) and (ii) that

$$\sum_{k\in\mathbb{Z}} c_k\overline{c_{k-2\ell}} = \begin{cases} 1 & \text{for}\ell = 0 \\ 0 & \text{for } \ell \neq 0 \end{cases}$$

and we know

$$\sum_k c_k = \sqrt{2}$$

from 1.5 (iv).

From Lemma 1.13 we know that $p(\xi) = \sum_{k\in\mathbb{Z}} c_k e^{-2\pi i k\xi}$ satisfies

$$|p(\xi)|^2 + |p(\xi + 1/2)|^2 = 2.$$

If we take $\xi = 0$, we have

$$\begin{aligned} p(0) &= \sum_k c_k = \sqrt{2} \\ |p(0)|^2 + |p(1/2)|^2 &= 2 + |p(1/2)|^2 = 2 \end{aligned}$$

and so $p(1/2) = 0$. That is

$$\begin{aligned} \sum_{k\in\mathbb{Z}} c_k e^{-2\pi i k(1/2)} &= 0 \\ \sum_{k\in\mathbb{Z}} c_k e^{-\pi i k} &= 0 \\ \sum_{k\in\mathbb{Z}} c_k (-1)^k &= 0 \text{ using } e^{-\pi i} = -1 \\ \sum_{k \text{ even}} c_k - \sum_{k \text{ odd}} c_k &= 0 \end{aligned}$$

Corollary 1.20 *Assuming ϕ satisfies the same hypotheses as in Proposition 1.19 and that*

$$\psi = \sum_k (-1)^k \overline{c_{1-k}} D_2 T_k \phi$$

then $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 0.$$

Proof. First note that $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ as it is a finite linear combination of functions $D_2 T_k \phi$ each in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

$$
\begin{aligned}
\int_{-\infty}^{\infty} \psi(x)\,dx &= \sum_k (-1)^k \overline{c_{1-k}} \int_{-\infty}^{\infty} (D_2 T_k \phi)(x)\,dx \\
&= \sum_k (-1)^k \overline{c_{1-k}} \int_{-\infty}^{\infty} \sqrt{2}\phi(2x-k)\,dx \\
&\quad \text{put } y = 2x - k \\
&\quad\ dy = 2dx \\
&= \left(\sum_k (-1)^k \overline{c_{1-k}}\right) \int_{-\infty}^{\infty} \sqrt{2}\phi(y)\,\frac{dy}{2} \\
\text{But} \\
\sum_k (-1)^k \overline{c_{1-k}} &= \sum_{k \text{ even}} \overline{c_{1-k}} - \sum_{k \text{ odd}} \overline{c_{1-k}} \\
&= \sum_{\ell \text{ odd}} \overline{c_\ell} - \sum_{\ell \text{ even}} \overline{c_\ell} \\
&= 0
\end{aligned}
$$

by 1.19. □

Proposition 1.21 *Assume now that ϕ is a continuous compactly supported solution of a finite dilation equation*

$$\phi = \sum_{k=k_0}^{k_1} c_k D_2 T_k \phi,$$

and that it is the scaling function for a multiresolution analysis.

Then

(i) $\sum_k \phi(x-k)$ *is a nonzero constant (and in fact the constant is* $\int_{-\infty}^{\infty} \phi(x)\,dx$ *and so is of modulus 1).*

(ii) If $\sum_{k \text{ even}} kc_k = \sum_{k \text{ odd}} kc_k$*, then there are coefficients* a_k *so that*

$$\sum_{k \in \mathbb{Z}} a_k \phi(x-k) = x.$$

In fact

$$a_k = \int_{-\infty}^{\infty} t\overline{\phi(t)}\,dt + k \int_{-\infty}^{\infty} \overline{\phi(t)}\,dt$$

Proof.

(i) We know that

$$\left| \int_{-\infty}^{\infty} \phi(x)\,dx \right| = 1$$

by Lemma 1.15. Thus if it is possible to find coefficients $(b_k)_{k\in\mathbb{Z}}$ so that

$$\sum_{k\in\mathbb{Z}} b_k \phi(x-k) = 1$$

(note that since ϕ is compactly supported in $[k_0, k_1]$ by Lemma 1.6, for each x there are only a finite number of k with $x - k \in [k_0, k_1]$ and so the sum has only a finite number of nonzero terms for each x) then

$$\begin{aligned}
\int_{-\infty}^{\infty} \overline{\phi(x)}\,dx &= \int_{-\infty}^{\infty} \overline{\phi(x-\ell)}\,dx \\
&= \int_{-\infty}^{\infty} \left(\sum_{k\in\mathbb{Z}} b_k \phi(x-k) \right) \overline{\phi(x-\ell)}\,dx \\
&= \int_{-\infty}^{\infty} \sum_{k\in\mathbb{Z}} b_k \phi(x-k)\overline{\phi(x-\ell)}\,dx \\
&= \int_{-\infty}^{\infty} \sum_{k=k_0-k_1+\ell}^{k_1-k_0+\ell} b_k \phi(x-k)\overline{\phi(x-\ell)}\,dx \\
&\quad \text{since } \phi(x-k) \neq 0 \Rightarrow x-k \in [k_0, k_1] \\
&\quad\quad \Rightarrow x \in [k_0+k, k_1+k] \\
&\quad \text{and so } \phi(x-k)\overline{\phi(x-\ell)} \neq 0 \\
&\quad\quad \Rightarrow [k_0+k, k_1+k] \cap [k_0+\ell, k_1+\ell] \neq \emptyset \\
&\quad\quad \Rightarrow k_0+k \leq k_1+\ell, \quad k_1+k \geq k_0+\ell \\
&\quad\quad \Rightarrow k \leq k_1-k_0+\ell, \quad k \geq k_0-k_1+\ell \\
&= \sum_{k=k_0-k_1+\ell}^{k_1-k_0+\ell} b_k \int_{-\infty}^{\infty} \phi(x-k)\overline{\phi(x-\ell)}\,dx \\
&= b_\ell
\end{aligned}$$

Thus there is only one possible choice for the b_k, but that does not show any such b_k exist. It does follow that

$$\sum_{k\in\mathbb{Z}} \phi(x-k) = \int_{-\infty}^{\infty} \phi(t)\,dt$$

if any such b_k exist (because the reciprocal of the integral is its complex conjugate).

To prove that this is possible, take

$$\lambda = \sum_{k \in \mathbb{Z}} \phi(k)$$

and

$$f(x) = \sum_{k \in \mathbb{Z}} \phi(x-k).$$

Then $f(x)$ is a continuous function because there are only finitely many nonzero terms in the sum at (or near) any given x. In fact for x in any interval $(n-1, n+1)$,

$$f(x) = \sum_{k=n-1-k_1}^{n+1-k_0} \phi(x-k)$$

is given by a finite sum of continuous functions and therefore is continuous on $(n-1, n+1)$. As this is so for all n, $f(x)$ is continuous on $\mathbb{R}$.

At $x = n \in \mathbb{Z}$,

$$f(n) = \sum_{k \in \mathbb{Z}} \phi(n-k) = \sum_{\ell \in \mathbb{Z}} \phi(\ell) = \lambda.$$

By induction on m, we show that for $m = 0, 1, 2, \ldots$ we have

$$f\left(\frac{n}{2^m}\right) = \lambda$$

Assuming that this is known to be true for m, consider

$$\begin{aligned}
f\left(\frac{n}{2^{m+1}}\right) &= \sum_{k \in \mathbb{Z}} \phi\left(\frac{n}{2^{m+1}} - k\right) \\
&= \sum_{k \in \mathbb{Z}} \sum_{j=k_0}^{k_1} c_j \sqrt{2} \phi\left(\frac{n}{2^m} - 2k - j\right) \\
& \quad \text{put } \ell = 2k + j \\
& \quad \quad j = \ell - 2k \\
&= \sum_{k \in \mathbb{Z}} \sum_{\ell=k_0+2k}^{k_1+2k} c_{\ell-2k} \sqrt{2} \phi\left(\frac{n}{2^m} - \ell\right) \\
&= \sum_{\ell} \sum_{k \in \mathbb{Z}} c_{\ell-2k} \sqrt{2} \phi\left(\frac{n}{2^m} - \ell\right)
\end{aligned}$$

$$= \sum_{\ell} \phi\left(\frac{n}{2^m} - \ell\right)$$

since

$$\begin{aligned} \textstyle\sum_{k\in\mathbb{Z}} c_{\ell-2k}\sqrt{2} &= \begin{cases} \sum_{r \text{ even}} c_r\sqrt{2} & \text{if } \ell \text{ even} \\ \sum_{r \text{ odd}} c_r\sqrt{2} & \text{if } \ell \text{ odd} \end{cases} \\ &= \tfrac{1}{2}\textstyle\sum_r c_r\sqrt{2} \\ &= \tfrac{1}{2}2 = 1 \end{aligned}$$

$$= f\left(\frac{n}{2^m}\right) = \lambda$$

This completes the induction proof and so we can conclude from continuity of $f(x)$ and density of the rationals of the form $n/2^m$ in $\mathbb{R}$ that $f(x) \equiv \lambda$.

From the remarks at the beginning of the proof we could conclude that

$$1 = \int_{-\infty}^{\infty} \sum_k \phi(x-k)\overline{\phi(x)}\,dx = \int_{-\infty}^{\infty} f(x)\overline{\phi(x)}\,dx = \lambda \int_{-\infty}^{\infty} \overline{\phi(x)}\,dx$$

and thus $\lambda = \int_{-\infty}^{\infty} \phi(x)\,dx$.

(ii) Observe that if its is possible to find $(a_k)_{k\in\mathbb{Z}}$ so that

$$\sum_{k\in\mathbb{Z}} a_k \phi(x-k) = x$$

then we can calculate a_ℓ as

$$\begin{aligned} \int_{-\infty}^{\infty} x\overline{\phi(x-\ell)}\,dx &= \sum_{k\in\mathbb{Z}} a_k \int_{-\infty}^{\infty} \phi(x-k)\overline{\phi(x-\ell)}\,dx \\ &= a_\ell \end{aligned}$$

and so

$$\begin{aligned} a_\ell &= \int_{-\infty}^{\infty} x\overline{\phi(x-\ell)}\,dx \\ &\quad \text{put } y = x-\ell \\ &= \int_{-\infty}^{\infty} (y+\ell)\overline{\phi(y)}\,dx \\ &= \int_{-\infty}^{\infty} y\overline{\phi(y)}\,dx + \ell \int_{-\infty}^{\infty} \overline{\phi(y)}\,dx \end{aligned}$$

To prove that any such a_k exist, we let

$$\lambda = \sum_{k\in\mathbb{Z}} \phi(k) = \int_{-\infty}^{\infty} \phi(x)\,dx$$

(from part (i)). We know that $|\lambda| = 1$ and so $1/\lambda = \overline{\lambda}$. Replacing $\phi(x)$ by $\overline{\lambda}\phi(x)$ we can assume that $\lambda = 1$.

Then put

$$\mu = \sum_{k \in \mathbb{Z}} k\phi(k)$$

(a finite sum because ϕ is compactly supported) and

$$g(x) = \sum_{k \in \mathbb{Z}} (\mu + k)\phi(x - k)$$

We aim to show that $g(x) \equiv x$.

First note that $g(x)$ is continuous (by the same sort of argument as was used to show that $f(x)$ was continuous in (i): the $g(x)$ is a finite sum of continuous terms $(\mu + k)\phi(x - k)$ on any interval $(n - 1, n + 1)$).

Next observe that for $\ell \in \mathbb{Z}$ we have

$$\begin{aligned} g(\ell) &= \sum_k (\mu + k)\phi(\ell - k) \\ &\quad \text{put } m = \ell - k \\ &\qquad\quad k = \ell - m \\ &= \sum_m (\mu + \ell - m)\phi(m) \\ &= \sum_m (\mu + \ell)\phi(m) + \sum_m (-m)\phi(m) \\ &= (\mu + \ell)\sum_m \phi(m) - \sum_m m\phi(m) \\ &= \mu + \ell - \mu \\ &= \ell \end{aligned}$$

Next we show by induction on n that

$$g\left(\frac{\ell}{2^n}\right) = \frac{\ell}{2^n} \quad \forall \ell \in \mathbb{Z}, n = 0, 1, 2, \ldots$$

The initial step $n = 0$ has just been done and it remains to do the induction step. Assume the result is true for n and look at

$$g\left(\frac{\ell}{2^{n+1}}\right) = \sum_k (\mu + k)\phi\left(\frac{\ell}{2^{n+1}} - k\right)$$

$$
\begin{aligned}
&= \sum_k (\mu+k) \sum_j c_j \sqrt{2}\phi\left(\frac{\ell}{2^n} - 2k - j\right) \\
&\quad \text{put } m = 2k+j \\
&\quad\quad j = m - 2k \\
&= \sum_k (\mu+k) \sum_m c_{m-2k} \sqrt{2}\phi\left(\frac{\ell}{2^n} - m\right) \\
&= \sum_m \left(\sum_k (\mu+k) c_{m-2k}\sqrt{2}\right)\phi\left(\frac{\ell}{2^n} - m\right)
\end{aligned}
$$

Looking at the bracketed sum, we find

$$
\begin{aligned}
&\sum_k (\mu+k) c_{m-2k}\sqrt{2} \\
&= \sum_k \left(\mu + \frac{m}{2} - \frac{m}{2} + k\right) c_{m-2k}\sqrt{2} \\
&= \sum_k \left(\mu + \frac{m}{2}\right) c_{m-2k}\sqrt{2} - \sum_k \left(\frac{m}{2} - k\right) c_{m-2k}\sqrt{2} \\
&= \left(\mu + \frac{m}{2}\right) \sum_k c_{m-2k}\sqrt{2} - \frac{1}{2}\sum_k (m-2k) c_{m-2k}\sqrt{2} \\
&= \left(\mu + \frac{m}{2}\right) 1 - \frac{1}{2}\sqrt{2} \sum_{r-m \text{ even}} r c_r \\
&\quad \text{see the proof of (i) for the justification of the 1.}
\end{aligned}
$$

But

$$
\sum_{r-m \text{ even}} r c_r = \begin{cases} \sum_{r \text{ even}} r c_r & \text{if } m \text{ even} \\ \sum_{r \text{ odd}} r c_r & \text{if } m \text{ odd} \end{cases}
$$

and we have assumed

$$
\sum_{r \text{ even}} r c_r = \sum_{r \text{ odd}} r c_r
$$

and so we conclude that both sums

$$
\sum_{r \text{ even}} r c_r = \sum_{r \text{ odd}} r c_r = \frac{1}{2}\sum_r r c_r.
$$

Now put $A = \sum_r r c_r$ and then we have

$$
\sum_k (\mu+k) c_{m-2k} = \mu + \frac{m}{2} - \frac{\sqrt{2}}{2}\frac{A}{2} = \mu + \frac{m}{2} - \frac{\sqrt{2}A}{4}.
$$

Using this in the above calculation, we have

$$\begin{aligned}
g\left(\frac{\ell}{2^{n+1}}\right) &= \sum_m \left(\mu + \frac{m}{2} - \frac{\sqrt{2}A}{4}\right)\phi\left(\frac{\ell}{2^n} - m\right) \\
&= \sum_m \left(\frac{\mu}{2} + \frac{m}{2}\right)\phi\left(\frac{\ell}{2^n} - m\right) \\
&\quad + \sum_m \left(\frac{\mu}{2} - \frac{\sqrt{2}A}{4}\right)\phi\left(\frac{\ell}{2^n} - m\right) \\
&= \frac{1}{2}g\left(\frac{\ell}{2^n}\right) + \left(\frac{\mu}{2} - \frac{\sqrt{2}A}{4}\right)\sum_m \phi\left(\frac{\ell}{2^n} - m\right) \\
&= \frac{1}{2}g\left(\frac{\ell}{2^n}\right) + \left(\frac{\mu}{2} - \frac{\sqrt{2}A}{4}\right)1 \\
&= \frac{1}{2}\frac{\ell}{2^n} + \left(\frac{\mu}{2} - \frac{\sqrt{2}A}{4}\right) \\
&\quad \text{using the induction hypothesis} \\
&= \frac{\ell}{2^{n+1}} + \left(\frac{\mu}{2} - \frac{\sqrt{2}A}{4}\right)
\end{aligned}$$

We want the bracketed term to be zero and we can show this by applying the above to $\ell = 2s$ even. We get

$$g\left(\frac{2s}{2^{n+1}}\right) = g\left(\frac{s}{2^n}\right) = \frac{2s}{2^{n+1}} + \left(\frac{\mu}{2} - \frac{\sqrt{2}A}{4}\right) = \frac{s}{2^n} + \left(\frac{\mu}{2} - \frac{\sqrt{2}A}{4}\right)$$

but we know that

$$g\left(\frac{s}{2^n}\right) = \frac{s}{2^n}$$

and so we must have $\left(\frac{\mu}{2} - \frac{\sqrt{2}A}{4}\right) = 0$. Thus the induction step is complete.

(It is not of interest now, but we have actually shown

$$\mu = \frac{\sqrt{2}A}{2} = \frac{1}{2}\sum_r rc_r\sqrt{2}$$

which must then be equal to $\sum_k k\phi(k)$, the definition of μ.)

Having established $g\left(\frac{\ell}{2^n}\right) = \frac{\ell}{2^n}$ (for all $\ell \in \mathbb{Z}, n \geq 0$) it follows by continuity that $g(x) \equiv x$, or

$$\sum_k (\mu + k)\phi(x - k) \equiv x.$$

Corollary 1.22 *With the same hypotheses as in 1.21 and*

$$\psi = \sum_k (-1)^k \overline{c_{1-k}} d_2 T_k \phi$$

then

$$\int_{-\infty}^{\infty} (\alpha x + \beta)\psi(x - j)\, dx = 0$$

for all $\alpha, \beta \in \mathbb{C}$, *all* $j \in \mathbb{Z}$.

Proof. From 1.21 we know that there are a_k and b_k so that

$$\begin{aligned} \sum_{k \in \mathbb{Z}} b_k \phi(x - k) &= 1 \\ \sum_{k \in \mathbb{Z}} a_k \phi(x - k) &= x \end{aligned}$$

and hence

$$\sum_{k \in \mathbb{Z}} (\alpha a_k + \beta b_k)\phi(x - k) = \alpha x + \beta$$

We deduce that

$$\begin{aligned} \int_{-\infty}^{\infty} (\alpha x + \beta)\overline{\psi(x - j)}\, dx &= \int_{-\infty}^{\infty} \sum_{k \in \mathbb{Z}} (\alpha a_k + \beta b_k)\phi(x - k)\overline{\psi(x - j)}\, dx \end{aligned}$$

Note that $\phi(x - k)$ is supported in support$(\phi) + k$ while $\psi(x - j)$ has compact support and so there are only a finite number of terms in the sum which can ever be nonzero

$$\begin{aligned} &= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} (\alpha a_k + \beta b_k)\phi(x - k)\overline{\psi(x - j)}\, dx \\ &= 0 \text{ since } T_k\phi \perp T_j\psi \end{aligned}$$

It follows that

$$\int_{-\infty}^{\infty} (\alpha x + \beta)\psi(x - j)\, dx = \overline{\int_{-\infty}^{\infty} (\overline{\alpha} x + \overline{\beta})\overline{\psi(x - j)}\, dx} = 0.$$

Corollary 1.23 *With the same hypotheses as in 1.22, if $f(x) = \alpha x + \beta$ for all x in the support of $D_{2^n}T_j\psi$, and $f \in L^2(\mathbb{R})$, then*

$$f \perp D_{2^n}T_j\psi$$

Proof.

$$\begin{aligned}
\langle f, D_{2^n}T_j\psi &= \int_{-\infty}^{\infty} f(x)\overline{(D_{2^n}T_j\psi)(x)}\,dx \\
&= \int_{-\infty}^{\infty} (\alpha x + \beta)\overline{(D_{2^n}T_j\psi)(x)}\,dx \\
&= \int_{-\infty}^{\infty} (\alpha x + \beta)\sqrt{2^n}\overline{\psi(2^n x - j)}\,dx \\
&\quad \text{put } y = 2^n x \\
&\quad\ \ dy = 2^n\,dx \\
&= \int_{-\infty}^{\infty} \left(\alpha\frac{y}{2^n} + \beta\right)\overline{\psi(y-j)}\,\frac{dy}{2^n} \\
&= 0 \text{ by } 1.22
\end{aligned}$$

Note. This Corollary tells us that in a wavelet expansion

$$f = \sum_{n,j\in\mathbb{Z}} \langle f, D_{2^n}T_j\psi\rangle D_{2^n}T_j\psi$$

of $f \in L^2(\mathbb{R})$ we get zero contributions from certain n and j where there is a linear part of the graph of f. But to make this work we need ψ to come from a multiresolution analysis with a scaling function ϕ which is compactly supported and continuous, and we need $\sum_{k \text{ even}} kc_k = \sum_{k \text{ odd}} kc_k$.

A Extension of the Fourier transform to $L^2(\mathbb{R})$

The integral formula for the Fourier transform

$$\mathcal{F}f(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx$$

makes good sense only for $f \in L^1(\mathbb{R})$, but we can extend $\mathcal{F}$ to $L^2(\mathbb{R})$ via a continuity argument.

One needs to establish that for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we have $\mathcal{F}f \in L^2(\mathbb{R})$ and

$$\|\mathcal{F}f\|_2 = \|f\|_2.$$

Using linearity of $\mathcal{F}$ this implies that the restriction of $\mathcal{F}$ to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$\mathcal{F}_{\text{restricted}} \colon L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

is uniformly continuous. In fact $f_1, f_2 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \Rightarrow$

$$\|\mathcal{F}(f_1) - \mathcal{F}(f_2)\|_2 = \|\mathcal{F}(f_1 - f_2)\|_2 = \|f_1 - f_2\|_2$$

and so, given any $\varepsilon > 0$, $\|f_1 - f_2\|_2 < \delta = \varepsilon \Rightarrow \|\mathcal{F}(f_1) - \mathcal{F}(f_2)\|_2 < \varepsilon$.

Now $\mathcal{F}_{\text{restricted}}$ is a uniformly continuous function on a dense subset $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ of $L^2(\mathbb{R})$ with values in a complete space $L^2(\mathbb{R})$, and from this there is a general theorem that states that $\mathcal{F}_{\text{restricted}}$ has a unique continuous extension to a continuous

$$\mathcal{F}_{\text{extended}} \colon L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

Moreover, it is easy to see that this $\mathcal{F}_{\text{extended}}$ is linear.

The general way to define $\mathcal{F}_{\text{extended}} f$ for $f \in L^2(\mathbb{R})$ is to take any sequence $(f_n)_{n=1}^\infty$ in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ which converges to f in $L^2(\mathbb{R})$. Then $(f_n)_{n=1}^\infty$ is a Cauchy sequence in $L^2(\mathbb{R})$ and by uniform continuity $\left(\mathcal{F}_{\text{restricted}} f_n\right)_{n=1}^\infty$ must be Cauchy in $L^2(\mathbb{R})$. Hence $\lim_{n\to\infty} \mathcal{F}_{\text{restricted}} f_n$ must exist in $L^2(\mathbb{R})$ by completeness and we define $\mathcal{F}_{\text{extended}} f$ to be that limit. The thing we need to check is that we only get one value for $\mathcal{F}_{\text{extended}} f$ in this way, that is that if $(g_n)_{n=1}^\infty$ is another sequence in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ which converges to f in $L^2(\mathbb{R})$, then

$$\lim_{n\to\infty} \mathcal{F}_{\text{restricted}} f_n = \lim_{n\to\infty} \mathcal{F}_{\text{restricted}} g_n \tag{2}$$

but this can be verified by interspersing the two sequences to consider the sequence

$$(f_1, g_1, f_2, g_2, \ldots) \to f \text{ in } L^2(\mathbb{R}).$$

Since

$$(\mathcal{F}_{\text{restricted}} f_1, \mathcal{F}_{\text{restricted}} g_1, \mathcal{F}_{\text{restricted}} f_2, \mathcal{F}_{\text{restricted}} g_2, \ldots)$$

must have a limit, we get (2).

A more concrete way to say what the extension is is to choose the sequence $(f_n)_{n=1}^\infty$ explicitly to be

$$f_n(x) = \chi_{[-n,n]}(x) f(x) = \begin{cases} f(x) & \text{if } |x| \leq n \\ 0 & \text{if } |x| > n \end{cases}$$

Then

$$\begin{aligned}
\|f - f_n\|_2^2 &= \int_{-\infty}^{\infty} |f_n(x) - f(x)|^2 \, dx \\
&= \int_{-\infty}^{\infty} \left(\chi_{(-\infty,-n)}(x) + \chi_{(n,\infty)}(x)\right) |f(x)|^2 \, dx \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

by the Lebesgue dominated convergence theorem (because the integrand $\to 0$ pointwise as $n \to \infty$ and is pointwise dominated in absolute value by $|f(x)|^2$ which is integrable). Note that $f_n \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ as it is easily seen to be in $L^2(\mathbb{R})$ and it is compactly supported (on $[-n, n]$ and so $\|f_n\|_1 \leq \sqrt{2n}\|f_n\|_2 \leq \sqrt{2n}\|f\|_2$ — see proof of part (iv) of Proposition 1.5). Thus we can say explicitly that

$$\mathcal{F}_{\text{extended}} f = \lim_{n\to\infty} \mathcal{F} f_n$$

(limit in $L^2(\mathbb{R})$ norm). For $\mathcal{F} f_n$ we have the integral formula

$$\begin{aligned}
\mathcal{F} f_n(\xi) &= \int_{-\infty}^{\infty} f_n(x) e^{-2\pi i x \xi} \, dx \\
&= \int_{-n}^{n} f(x) e^{-2\pi i x \xi} \, dx
\end{aligned}$$

The extension of $\mathcal{F}$ to $L^2(\mathbb{R})$ will satisfy

$$\left\|\mathcal{F}_{\text{extended}} f\right\|_2 = \|f\|_2$$

and we usually denote the extension by $\mathcal{F}$.

An important fact about this extension is that it has an inverse (so it is a bijection of $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$) and the inverse mapping is given by almost the same formula as $\mathcal{F}$. The inverse will be the extension (in the same way as $\mathcal{F}$ is extended) of the mapping on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ given by

$$(\mathcal{F}^{-1} g)(x) = \int_{-\infty}^{\infty} g(\xi) e^{2\pi i x \xi} \, d\xi$$