Mathematics 414 2003–04
Exercises 6
[Due Monday March 1st, 2004.]

1. Let $(f_n)_{n=1}^\infty$ be a sequence in $C(G)$ (where $G \subset \mathbb{C}$ is open), $f\colon G \to \mathbb{C}$ a function and $K \subset G$ compact. Verify carefully that $f_n \to f$ uniformly on K (as $n \to \infty$) if and only if
$$d_K(f_n, f) = \sup_{z \in K} |f_n(z) - f(z)| \to 0$$
as $n \to \infty$.

2. For each of the following open $G \subset \mathbb{C}$, exhibit explicitly an exhaustive sequence $(K_n)_{n=1}^\infty$ of compact subsets of G.
   (a) $G = D(a, r)$
   (b) $G = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$
   (c) $G = \mathbb{C} \setminus \mathbb{Z}$
   (d) $G = \{z \in \mathbb{C} : 0 < \Re z < 1\}$

3. Let $G \subset \mathbb{C}$ be open, $(f_n)_{n=1}^\infty$ a sequence of functions $f_n\colon G \to \mathbb{C}$ and $f\colon G \to \mathbb{C}$ a function. We say that $f_n \to f$ *locally uniformly* on G if for each $z_0 \in G$ there is a disk $D(z_0, \delta)$ about z_0 (of positive radius $\delta > 0$) so that $f_n \to f$ uniformly on $D(z_0, \delta)$.

   Show that $f_n \to f$ locally uniformly if and only if $f_n \to f$ uniformly on compact subsets of G.

4. Let $G \subset \mathbb{C}$ be open and ρ a metric on $H(G)$ such that convergence in ρ corresponds to uniform convergence on compact subsets of G. Fix $z_0 \in G$. Show that that the point evaluation map $\delta_{z_0}\colon H(G) \to \mathbb{C}$ given by $\delta_{z_0}(f) = f(z_0)$ is continuous.

   That is show that if $f_n \to f$ in $(H(G), \rho)$ then $\delta_{z_0}(f_n) \to \delta_{z_0}(f)$.

   Show that δ_{z_0} is linear and multiplicative ($\delta_{z_0}(fg) = \delta_{z_0}(f)\delta_{z_0}(g)$). Deduce that $\mathcal{I} = \ker \delta_{z_0} = \{f \in H(G) : \delta_{z_0}(f) = 0\}$ is a closed ideal in $H(G)$ (that is $\mathcal{I}$ is a vector subspace, has the property $fg \in \mathcal{I}$ if $f \in \mathcal{I}$, $g \in H(G)$ and is also closed as a subset of $H(G)$).

5. Let $G \subset \mathbb{C}$ be open and ρ a metric on $H(G)$ such that convergence in ρ corresponds to uniform convergence on compact subsets of G.
   (a) Show that the map $f \mapsto f' : H(G) \to H(G)$ is continuous.
   That is show that if $f_n \to f$ in $(H(G), \rho)$ then $f_n' \to f'$ in $(H(G), \rho)$. [Hint: Use the Cauchy integral formula to show local uniform convergence.]
   (b) Show that if $\mathcal{F} \subset H(G)$ is relatively compact, then $\{f' : f \in \mathcal{F}\}$ is also relatively compact in $H(G)$.