Mathematics 414 2003–04
Exercises 5
[Due Monday February 16th, 2004.]

1. Suppose f and g are meromorphic functions on an open set $G \subseteq \mathbb{C}$, and let H_f be the points where f is analytic in G ($H_f = G \setminus \{\text{poles of } f\}$), H_g the set where g is analytic. Show that in each of the following cases, there is a unique meromorphic function h on G such that:

   (a) $h(z) = f(z) + g(z)$ for $z \in H_f \cap H_g$. [We use this as a definition of $f + g$.]

   (b) $h(z) = f(z)g(z)$ for $z \in H_f \cap H_g$. [We use this as a definition of the product fg.]

   If λf is defined to be gf with $g(z) = \lambda$ the constant function, show that $M(G) = \{f : f \text{ meromorphic on } G\}$ is an algebra over $\mathbb{C}$ with an identity (under the above addition and multiplication operations).

2. Show that if f is analytic on an open set that includes the closed unit disk $\overline{D}(0,1)$ and if $f(\{z \in \mathbb{C} : |z| = 1\}) \subset D(0,1)$, then f has exactly one fixed point in $D(0,1)$. [A *fixed point* is a point z where $f(z) = z$.]

3. How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the disc $|z| < 1$? [Hint: Look at the biggest term when $|z| = 1$ and apply Rouché's theorem.]

4. How many roots of the equation $z^4 - 6z + 3 = 0$ have modulus between 1 and 2?