Mathematics 414 2003–04
Exercises 4
[Due Monday February 2nd, 2004.]

1. Let γ be a piecewise C^1 closed curve in $\mathbb{C} \setminus \{i, -i\}$. Find all possible values for
$$\int_\gamma \frac{1}{z^2+1}\,dz$$

2. Let γ: $[0, 2\pi] \to \mathbb{C}$ be the curve $\gamma(t) = re^{it}$ where $r > 0$ and $r \neq 2$. Find
$$\int_\gamma \frac{z^2+1}{z(z^2+4)}\,dz$$
as a function of r.

3. Suppose $f(z)$ is analytic in an annulus $R_1 < |z-a| < R_2$ (where $0 \leq R_1 < R_2 \leq \infty$). Show that f has a Laurent series expansion
$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n \qquad (R_1 < |z-a| < R_2).$$

4. Suppose $f(z)$ is analytic in an annulus $R_1 < |z-a| < R_2$ (where $0 \leq R_1 < R_2 \leq \infty$) and that f has two Laurent series expansions
$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n = \sum_{n=-\infty}^{\infty} b_n(z-a)^n \qquad (R_1 < |z-a| < R_2).$$
Show that $a_n = b_n$ for all n. [Hint: Laurent series converge uniformly on $|z-a| = r$ for $R_1 < r < R_2$. Allows justification of exchange of integration and sum.]

   {**Note.** We have been using this quite often, though it was not stated in the notes. We did justify the power series version of this.}

5. Show that $f(z) = \exp(1/z^2)$ has an essential singularity at $z = 0$. Given $w \in \mathbb{C} \setminus \{0\}$, show that there is a sequence $(z_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} z_n = 0$ and $f(z_n) = w$.

6. Give an example of an analytic function that is analytic except at $z = 1$ and $z = 2$, has an essential singularity at $z = 1$ and a pole of order 3 at $z = 2$. Find its residue at $z = 2$.