Mathematics 414 2003–04
Exercises 3
[Due Monday January 5th, 2004.]

1. Let G denote either of $\mathbb{C} \setminus \{t : t \in \mathbb{R}, t \leq 0\}$ or $\mathbb{C} \setminus \{it : t \in \mathbb{R}, t \leq 0\}$. Show that there is a branch of $\log z$ in G (with the value 0 at $1 \in G$). Deduce that if $z \in \mathbb{C} \setminus \{0\}$ then there exists $w \in \mathbb{C}$ with $e^w = z$.

2. Let $G \subset \mathbb{C}$ be a nonempty connected open set and let $f \colon G \to \mathbb{C}$ be an analytic function which is never zero in G. If $\frac{f'(z)}{f(z)}$ has an antiderivative on G show that there is a branch of the logarithm of f on G.

3. Show that if $G \subset \mathbb{C}$ is connected and if each closed curve $\gamma \colon [\alpha, \beta] \to G$ in G is homotopic in G to a constant curve, then it is homotopic to the constant curve $\gamma(\alpha)$. [Hint: Starting with a homotopy $H(t, s)$ to a constant curve, put $H_1(t, s) = H(t, 2s)$ for $s \leq 1/2$ and $H_1((t, s) = H(\alpha, 1 - 2s)$ for $1/2 \leq s \leq 1$.]

4. A set $G \subset \mathbb{C}$ is called *star-shaped* with respect to a point $a \in G$ if for any $z \in G$ and any $0 < t < 1$ it is true that $ta + (1 - t)z \in G$. Show that star-shaped sets G are simply-connected.

5. Let f be analytic on a connected open set $G \subset \mathbb{C}$ and suppose that $a \in G$ is a local minimum point for $|f(z)|$. Show that $f(a) = 0$ or f is constant.

6. Let f be analytic on a connected open set $G \subset \mathbb{C}$ that includes the closed disc $\overline{D}(a, r)$ ($r > 0$) and suppose that the modulus $|f(z)|$ of f is constant on the circle $|z - a| = r$. Show that either $f(z) = 0$ for some $z \in D(a, r)$ of f is constant.

7. Let $f(z) = u(z) + iv(z)$ ($u(z) = \Re f(z)$, $v(z) = \Im f(z)$) be analytic on a connected open set $G \subset \mathbb{C}$.

   (a) Show that $u(z)$ and $v(z)$ are *harmonic functions* on G (that is they satisfy Laplace's equation $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x + iy) = 0$)

   (b) Show that if u is constant then so is f

   (c) Show that $u(z)$ has neither a local maximum nor a local minimum point in G, unless u is consant. [Hint: Apply the maximum modulus principle to $e^{f(z)}$.]

8. Let $f, g \colon D(0, 1) \to \mathbb{C}$ be analytic functions.

   (a) If $f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right)$ for each $n = 1, 2, 3 \ldots$, show that $f = g$.

   (b) Show that $f\left(\frac{1}{n}\right) = \frac{1}{\sqrt{n}}$ for each $n = 1, 2, 3 \ldots$ is impossible.