Mathematics 414 2003–04
Exercises 2
[Due Tuesday November 18th, 2003.]

1. Suppose $S \subseteq \mathbb{C}$ is any set and $f_n \colon S \to \mathbb{C}$ ($n = 1, 2, 3, \ldots$) are a sequence of continuous functions. Suppose $f_n \to f$ uniformly on S (as $n \to \infty$) for a function $f \colon S \to \mathbb{C}$. Suppose $\gamma \colon [a, b] \to S$ is a piecewise C^1 curve in S. Show that

$$\lim_{n \to \infty} \int_\gamma f_n(z)\, dz = \int_\gamma f(z)\, dz$$

2. Suppose $G \subseteq \mathbb{C}$ is an open set and $\phi \colon [a, b] \times G \to \mathbb{C}$ is a continuous function (on $[a, b] \times \mathbb{C} \subseteq \mathbb{R} \times \mathbb{C} \equiv \mathbb{R}^3$). Define a function $f \colon G \to \mathbb{C}$ by

$$f(z) = \int_{t=a}^{b} \phi(t, z)\, dt \qquad (z \in G).$$

Show that f is continuous on G. [Hint. Fix $z \in G$ and show how to choose $\delta_0 > 0$ so that $\overline{D}(z, \delta_0) \subseteq G$. Use uniform continuity of ϕ on the compact set $[a, b] \times \overline{D}(z, \delta_0)$.]

(This fact is used in the proof of Theorem 1.20, to show that Ind_γ is a continuous function on $\mathbb{C} \setminus \gamma$ when γ is a piecewise C^1 curve. We can write

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} \frac{\gamma'(t)}{\gamma(t) - z}\, dt$$

where $\gamma \colon [a, b] \to \mathbb{C}$, $a = a_0 < a_1 < \cdots < a_n = b$ and γ is C^1 on each interval $[a_{j-1}, a_j]$. Our exercise shows that each of the integrals is a continuous function of z.)

3. Suppose $G \subseteq \mathbb{C}$ is an open set and $f_n \colon G \to \mathbb{C}$ ($n = 1, 2, 3 \ldots$) are a sequence of analytic functions which converge uniformly on G to a function $f \colon G \to \mathbb{C}$.

   (a) Show that f must be analytic. [Hint. Morera's Theorem.]

   (b) Show that

   $$\lim_{n \to \infty} f_n'(z) = f'(z)$$

   for each $z \in G$. [Hint. Fix z and choose δ_0 as in question 2. Use the Cauchy integral formula to express $f_n'(z)$ as an integral around $|\zeta - z| = \delta_0$. Apply question 1.]

4. Suppose $f \colon \mathbb{C} \to \mathbb{C}$ is an entire function and it satisfies

$$|f(z)| \leq C(|z| + 1)^n \qquad (\text{all } z \in \mathbb{C}),$$

for some constant $C \geq 0$ and some nonnegative $n \in \mathbb{Z}$. Show that f must be a polynomial of degree at most n. [Hint: Consider the proof of Liouville's theorem.]

5. Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is an entire function which is not constant and let $w \in \mathbb{C}$, $\delta > 0$. Show that there is some $z \in \mathbb{C}$ with $|f(z) - w| < \delta$. [Hint. If not, apply Liouville's theorem to $g(z) = 1/(f(z) - w)$.]

   (Thus fact is called the Casorati-Weierstrass theorem. Notice that it says that $f(\mathbb{C})$ is dense in $\mathbb{C}$ for any nonconstant entire function f.)

6. Let $G \subseteq \mathbb{C}$ be open and let γ be a piecewise C^1 curve in G which is null-homotopic in G.

   (a) Show that $\text{Ind}_\gamma(a) = 0$ for each $a \in \mathbb{C} \setminus G$.

   (b) Let $a \in G \setminus \gamma$ and let $f\colon G \to \mathbb{C}$. Show that the Cauchy integral formula

   $$\frac{m!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^m}\, dz = f^{(m)}(a)\text{Ind}_\gamma(a)$$

   holds. [Hint: Let σ be the circle $|z - a| = r$ traversed $-\text{Ind}_\gamma(a)$ where $r > 0$ is suitably small so that $\sigma \subset G$. Consider the chain $\Gamma = \gamma + \sigma$ in $G \setminus \{a\}$.]