Mathematics 414 2003–04
Exercises 1
[Due Tuesday October 28th, 2003.]

1. Show that a set $S \subseteq \mathbb{C}$ fails to be connected if and only if it is possible to find a continuous function $f\colon S \to \mathbb{C}$ with range $f(S) = \{0, 1\}$.

2. Use the Intermediate Value Theorem to show that intervals in $\mathbb{R}$ are connected.

3. (a) Show that path-connected sets $S \subseteq \mathbb{C}$ are connected.
   (b) Show that connected open subsets $G \subseteq \mathbb{C}$ are path-connected, and that it is possible to join any two points in G using a continuous path made up of a finite number of line segments with each line segment parallel to the real or imaginary axis.

4. (a) Show that open discs $D(z_0, r) \subseteq \mathbb{C}$ are connected.
   (b) Show that the set $S = D(0, 1) \cup \{1\} \cup D(2, 1)$ is connected.

5. Let G be a region in $\mathbb{C}$ and $f\colon G \to \mathbb{C}$ an analytic function with $f'(z) \equiv 0$ for all $z \in G$. Show that f is constant.

   Give an example to show that this fails to be true if G is allowed to be a disconnected open subset of $\mathbb{C}$.

6. Let G be a region in $\mathbb{C}$ and $f, g\colon G \to \mathbb{C}$ two analytic functions with the same real part (that is, $\Re f(z) \equiv \Re g(z)$ for all $z \in G$). Show that $f(z) - g(z)$ is constant in G.

7. Find all possible analytic functions $f\colon \mathbb{C} \to \mathbb{C}$ (functions analytic on all of $\mathbb{C}$ are often called *entire functions*) which have real part

$$\Re f(x + iy) \equiv 2x^3 - 6xy^2 + x^2 - y^2 - y$$

8. Show that the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$$

   is 1 and investigate its convergence at the points $z = 1, -1, i$ and $-i$.

9. Find two examples of power series with radius of convergence 1, both centered at the origin, where in one case the series converges uniformly for $|z| \leq 1$ and in the other it does not. (Give justifications.)