Chapter 6: The metric space $M(G)$ and normal families

Course 414, 2003–04

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Remark 6.1 For $G \subset \mathbb{C}$ open, we recall the notation $M(G)$ for the set (algebra) of all meromorphic functions on G. We now consider convergence in $M(G)$ is a way analogous to what we did for $H(G)$ in Chapter 5.

Before we do that, we explain how meromorphic functions on G can be regarded as functions or maps from G to an extended complex plane $\mathbb{C} \cup \{\infty\}$ with one extra 'point at infinity' added.

While the process of adding a point at 'infinity' can be carried out very abstractly (Alexandroff one point compactification of a locally compact topological space), for the complex plane we can visualise it rather geometrically via stereographic projection.

Stereographic Projection 6.2 There is a transformation which maps the complex plane $\mathbb{C}$ bijectively to a sphere in $\mathbb{R}^3$ with one point removed. To explain it we consider $\mathbb{C}$ as embedded in space $\mathbb{R}^3$ in the most obvious way:

$$z = x + iy \in \mathbb{C} \mapsto (x, y, 0) \in \mathbb{R}^3 \quad (x, y \in \mathbb{R})$$

and we consider the unit sphere S^2 in $\mathbb{R}^3$

$$S^2 = \{(\xi, \eta, \zeta) \in \mathbb{R}^3 : |\xi|^2 + |\eta|^2 + |\zeta|^2 = 1\}.$$

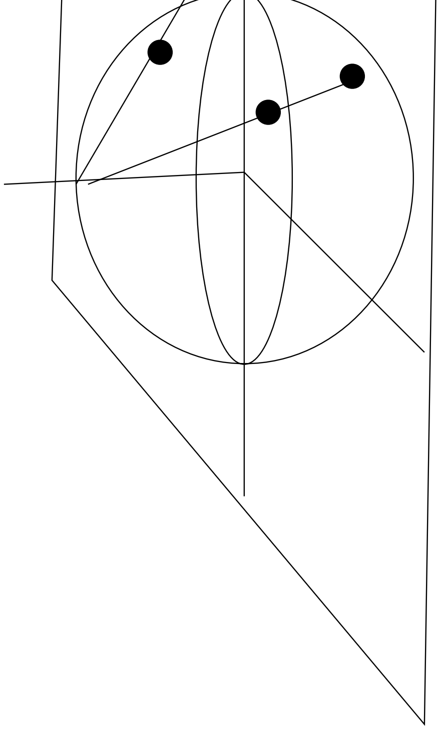
The 'North Pole' of the sphere $(0, 0, 1)$ will be a special point in our discussion and we sometimes write $\mathcal{NP}$ for $(0, 0, 1)$.

Define a mapping

$$P : \mathbb{C} \to S^2$$

by the geometrical rules that $P(z)$ is the point (other than the North Pole $\mathcal{NP}$) where the line joining $\mathcal{NP}$ to $z = (x, y, 0)$ intersects S^2. This map P is called *stereographic projection*.

From a picture you can see that if $|z| > 1$, then $P(z)$ is on the upper hemisphere. If $|z| = 1$, then $P(z)$ will be the 'same' as z or $(x, y, 0)$ while for $|z| < 1$, $P(z)$ will be in the lower hemisphere.



Proposition 6.3 *The stereographic projection map $P\colon \mathbb{C} \to S^2 \setminus \{\mathcal{NP}\} = S^2 \setminus \{(0,0,1)\}$ is a bijection and is given by*

$$P(z) = P(x+iy) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1} \right). \tag{1}$$

The inverse map is

$$P^{-1}(\xi, \eta, \zeta) = \left(\frac{\xi}{1-\zeta}, \frac{\eta}{1-\zeta}, 0 \right) \tag{2}$$

Proof. The line in $\mathbb{R}^3$ we used to define P has parametric equations

$$(\xi, \eta, \zeta) = (0,0,1) + t((x,y,0) - (0,0,1)) = (tx, ty, 1-t)$$

and this meets S^2 at the values of t where

$$\begin{aligned}
\xi^2 + \eta^2 + \zeta^2 &= 1 \\
t^2x^2 + t^2y^2 + (1-t)^2 &= 1 \\
t^2x^2 + t^2y^2 + 1 - 2t + t^2 &= 1 \\
t^2(x^2+y^2+1) - 2t &= 0 \\
t(t(x^2+y^2+1) - 2) &= 0
\end{aligned}$$

and so where $t = 0$ (the North Pole $\mathcal{NP}$) and where $t = 2/(x^2+y^2+1)$. This must be the value of t for the point $P(z)$, and so

$$\begin{aligned}
P(z) &= \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, 1 - \frac{2}{x^2+y^2+1} \right) \\
&= \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1} \right).
\end{aligned}$$

One can check in a straightforward way that the map $Q\colon S^2 \setminus \{\mathcal{NP}\} \to \mathbb{C}$ given by

$$Q(\xi, \eta, \zeta) = \left(\frac{\xi}{1-\zeta}, \frac{\eta}{1-\zeta}, 0 \right)$$

is the inverse by verifying $Q \circ P = \mathrm{id}_{\mathbb{C}}$ and $P \circ Q = \mathrm{id}$ are both identity maps.

Definition 6.4 *The* extended complex plane $\hat{\mathbb{C}}$ *defined as the complex plane* $\mathbb{C}$ *with one extra point, denoted* ∞*, adjoined.*

If $P(\infty)$ *is defined to be the North Pole* $\mathcal{NP} = (0,0,1)$ *in* S^2*, then stereographic projection* $P\colon \hat{\mathbb{C}} \to S^2$ *is a bijection.*

The sphere S^2 *(or* $\hat{\mathbb{C}}$ *which we can identify with* S^2 *via the stereographic projection map* P*) is called the* Riemann sphere.

For $z, w \in \hat{\mathbb{C}}$ *we introduce a notation* $\sigma(z,w)$ *for the Euclidean distance between* $P(z)$ *and* $P(w)$*. (This means the straight line distance in* $\mathbb{R}^3$*, as opposed to the length of the shortest path on the sphere.)* σ *is called the chordal distance or* chordal metric *on the Riemann sphere.*

Lemma 6.5 *For* $z, w \in \mathbb{C}$*,*

$$\begin{aligned}
\sigma(z,w) &= \frac{2|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}}\\
\sigma(z,\infty) &= \frac{2}{\sqrt{(1+|z|^2)}}
\end{aligned}$$

Also if z, w *are not zero*

$$\begin{aligned}
\sigma\left(\frac{1}{z},\frac{1}{w}\right) &= \sigma(z,w)\\
\sigma\left(\frac{1}{z},\infty\right) &= \sigma(z,0)
\end{aligned}$$

Proof. We have

$$P(x+iy) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right)$$

and we want to compute

$$\|P(x+iy) - P(r+is)\|$$

Observe that for unit vectors $v, w \in \mathbb{R}^3$

$$\|v-w\|^2 = (v-w).(v-w) = \|v\|^2 + \|w\|^2 - 2v.w = 2 - 2v.w$$

and so

$$\begin{aligned}
&\|P(x+iy) - P(r+is)\|^2(x^2+y^2+1)(r^2+s^2+1)\\
&\quad= 2\left[(x^2+y^2+1)(r^2+s^2+1) - (4xr+4ys+(x^2+y^2-1)(r^2+s^2-1))\right]\\
&\quad= 2[(x^2+y^2)(r^2+s^2) + (r^2+s^2) + (x^2+y^2) + 1\\
&\qquad -4(xr+ys) - (x^2+y^2)(r^2+s^2) + (r^2+s^2) + (x^2+y^2) - 1]\\
&\quad= 2\left[2(x^2+y^2) + 2(r^2+s^2) - 4(xr+ys)\right]\\
&\quad= 4[(x^2-2xr+r^2) + (y^2-2ys+s^2)]\\
&\quad= 4[(x-r)^2 + (y-s)^2]\\
&\quad= 4|(x+iy) - (r+is)|^2
\end{aligned}$$

Thus

$$\sigma(z,w)^2(|z|^2+1)(|w|^2+1) = 4|z-w|^2$$

and the formula for $\sigma(z,w)$ is as claimed.

All the other statements in the proposition are quite easy to check.

$$\begin{aligned}
\sigma(z,\infty)^2 &= \|P(x+iy)-(0,0,1)\|^2 \\
&= \left\|\left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}-1\right)\right\|^2 \\
&= \left\|\left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{-2}{x^2+y^2+1}\right)\right\|^2 \\
&= \frac{1}{(x^2+y^2+1)^2}(4x^2+4y^2+4) \\
&= \frac{4}{x^2+y^2+1} = \frac{4}{|z|^2+1} = \left(\frac{2}{\sqrt{1+|z|^2}}\right)^2
\end{aligned}$$

$$\sigma\left(\frac{1}{z},\frac{1}{w}\right) = \frac{2\left|\frac{1}{z}-\frac{1}{w}\right|}{\sqrt{\left(1+\frac{1}{|z|^2}\right)\left(1+\frac{1}{|w|^2}\right)}} = \frac{2|w-z|}{\sqrt{(|z|^2+1)(|w|^2+1)}} = \sigma(z,w)$$

where we multiplied above and below by $|zw| = \sqrt{|z|^2|w|^2}$. Finally,

$$\sigma\left(\frac{1}{z},\infty\right) = \frac{2}{\sqrt{\left(1+|\frac{1}{z}|^2\right)}} = \frac{2|z|}{\sqrt{(|z|^2+1)}} = \sigma(z,0).$$

Remark 6.6 Now $(\hat{\mathbb{C}},\sigma)$ is a metric space (in fact the 'same' as the sphere S^2 in $\mathbb{R}^3$ with the subset metric, or the distance from $\mathbb{R}^3$ restricted to S^2) and so we can look at open balls in $\hat{\mathbb{C}}$, open sets in $\mathbb{C}^2$, limits of sequences in $\mathbb{C}^2$, continuous functions and so on.

Proposition 6.7 *A subset $U \subset \hat{\mathbb{C}}$ is open in the metric space $(\hat{\mathbb{C}},\sigma) \iff$ is satisfies both*

(i) $U \cap \mathbb{C}$ is open (in $\mathbb{C}$ in the usual sense)

(ii) if $\infty \in U$ then there is some $r > 0$ so that

$$\{z \in \mathbb{C} : |z| > r\} \subset U$$

Proof. Note that the map $P|_{\mathbb{C}} : \mathbb{C} \to S^2 \setminus \{\mathcal{NP}\}$ is continuous because the formula (1) for P in Proposition 6.3 is clearly continuous from $\mathbb{R}^2$ to $\mathbb{R}^3$. Hence if U is open in $\hat{\mathbb{C}}$ then $P(U)$ is open in S^2 (because the metrics on $\hat{\mathbb{C}}$ and S^2 are copies of one another) and the inverse image $(P|_{\mathbb{C}})^{-1}(P(U)) = U \cap \mathbb{C}$ is therefore open in $\mathbb{C}$. If $\infty \in U$ ($U \subset \hat{\mathbb{C}}$ open), then U contains a ball of some positive radius $\delta > 0$ about ∞. That is

$$B_\sigma(\infty,\delta) = \{z \in \hat{\mathbb{C}} : \sigma(\infty,z) < \delta\} \subset U.$$

Assuming $\delta < 2$ (we can make δ smaller if necessary) we have for $z \in \mathbb{C}$

$$\sigma(\infty, z) = \frac{2}{\sqrt{1+|z|^2}} < \delta \iff 1 + |z|^2 > \frac{4}{\delta^2} \iff |z| > \sqrt{\frac{4}{\delta^2} - 1}$$

and so with $r = (1/\delta)\sqrt{4-\delta^2}$ we have

$$\{z \in \mathbb{C} : |z| > r\} \subset U$$

This was under the assumption $\infty \in U \subset \hat{\mathbb{C}}$ open.

Conversely assume $U \subset \hat{\mathbb{C}}$ satisfies the two conditions. The second condition tells us that ∞ is an interior point of U if it is in U at all.

$$\infty \in U \text{ and } \{z \in \mathbb{C} : |z| > r\} \subset U \Rightarrow B_\sigma(\infty, \delta) \subset U$$

with $\delta = 2/\sqrt{1+r^2}$.

The first condition $U \cap \mathbb{C}$ open in $\mathbb{C}$ plus continuity of $P^{-1} \colon S^2 \setminus \{\mathcal{NP}\} \to \mathbb{C}$ (which is clear from the formula (2) in Proposition 6.3) implies that

$$(P^{-1})^{-1}(U \cap \mathbb{C}) = P(U \cap \mathbb{C})$$

is open in $S^2 \setminus \{\mathcal{NP}\}$. It follows that every point of $P(U \cap \mathbb{C})$ is an interior point relative to S^2 and so (since P transforms the distance σ to the distances on S^2) all points of $U \cap \mathbb{C}$ are interior points (with respect to $(\hat{\mathbb{C}}, \sigma)$).

Definition 6.8 *If $G \subset \mathbb{C}$ is open and $f \colon G \to \hat{\mathbb{C}}$ is a function, then we say that f is analytic if it satisfies*

(a) f it is continuous (from G with its usual metric to $(\hat{\mathbb{C}}, \sigma)$);

(b)

$$f\,|_{f^{-1}(\mathbb{C})} \colon f^{-1}(\mathbb{C}) \to \mathbb{C}$$

is analytic (in the usual sense); [Note that $f^{-1}(\mathbb{C})$ is open in G by continuity of f, hence $f^{-1}(\mathbb{C})$ is open in $\mathbb{C}$ and so there is no problem looking at analyticity of this function.] and

(c) the restriction of $\dfrac{1}{f(z)}$ to $f^{-1}(\hat{\mathbb{C}}) \setminus \{0\}$ is analytic when we define $1/f(z) = 0$ at points where $f(z) = \infty$.

Note that $1/f(z)$ is continuous because of Lemma 6.5 and so the third condition means that f and $1/f$ are treated equally.

A reasonable way to look at it is that the two maps $z \in \hat{\mathbb{C}} \setminus \{\infty\} = \mathbb{C} \mapsto z \in \mathbb{C}$ and $w \in \hat{\mathbb{C}} \setminus \{0\} \mapsto 1/w \in \mathbb{C}$ (with $1/\infty$ interpreted as 0) are two coordinate charts on $\hat{\mathbb{C}}$. In this way we can regard $\hat{\mathbb{C}}$ as a 2-dimensional manifold[1]. Because the transition from coordinate z to coordinate w is given by an analytic transition function $w = 1/z$ on the intersection $\mathbb{C} \cap (\hat{\mathbb{C}} \setminus \{0\})$ where both coordinates can be used, and the opposite $w \mapsto z = 1/w$ is also analytic, we say that $\hat{\mathbb{C}}$ is a complex analytic manifold (of complex dimension 1, and these are actually called Riemann surfaces).

A map f with values in a complex manifold is analytic if f is continuous and each coordinate map composed with f is analytic (on the open set where it makes sense). Our definition of $f \colon G \to \hat{\mathbb{C}}$ analytic fits the pattern that is used for maps with values in a general Riemann surface.

Proposition 6.9 *If $G \subset \mathbb{C}$ is open and connected and $f \colon G \to \hat{\mathbb{C}}$ is analytic but not identically ∞, then the points of $f^{-1}(\infty)$ are isolated in G.*

That is $z_0 \in f^{-1}(\infty)$ implies there is a punctured disc $D(z_0, r) \setminus \{z_0\} \subset G \setminus f^{-1}(\infty)$.

Proof. Note that $f^{-1}(\hat{\mathbb{C}} \setminus \{0\})$ is open and $1/f$ is analytic there. If $z_0 \in f^{-1}(\infty)$, then $z_0 \in f^{-1}(\hat{\mathbb{C}} \setminus \{0\})$ and so there is $r > 0$ with $D(z_0, r) \subset f^{-1}(\hat{\mathbb{C}} \setminus \{0\})$ and $1/f$ is analytic on $D(z_0, r)$. Notice $1/f(z_0) = 1/\infty = 0$. If z_0 is not an isolated point of $f^{-1}(\infty)$ then z_0 is not an isolated zero of $1/f$ in $D(z_0, r)$. Then $1/f$ is identically 0 on $D(z_0, r)$ by the identity theorem for analytic function, and in fact $1/f$ is identically 0 on the connected component of z_0 in $f^{-1}(\hat{\mathbb{C}} \setminus \{0\})$. Call this connected component G_0. Then $f \equiv \infty$ on G_0 and (because connected components are open and closed) G_0 is open in $f^{-1}(\hat{\mathbb{C}} \setminus \{0\})$, hence in G.

But G_0 is also closed in G because if $(z_n)_{n=1}^\infty$ is a sequence in G_0 converging to a limit $z \in G$, then continuity of f implies $f(z) = \lim_{n\to\infty} f(z_n) = \infty$. So $z \in f^{-1}(\hat{\mathbb{C}} \setminus \{0\})$. Since G_0 is closed in $f^{-1}(\hat{\mathbb{C}} \setminus \{0\})$ (being a connected component), we have $z \in G_0$.

Thus $z_0 \in G_0 \subset G$ is open in G, closed in G and nonempty. As G is connected, $G_0 = G$ and so $f \equiv \infty$. But this contradicts the assumptions.

Theorem 6.10 *If $G \subset \mathbb{C}$ is open and f is a meromorphic function on G, then we can define an analytic $\hat{\mathbb{C}}$-valued function $f \colon G \to \hat{\mathbb{C}}$ be setting $f(z) = \infty$ at poles of f.*

Conversely, if $G \subset \mathbb{C}$ is open and connected and $f \colon G \to \hat{\mathbb{C}}$ is analytic but not identically ∞, then f is a meromorphic function on G with poles at the points in $f^{-1}(\infty)$.

Proof. Starting with f meromorphic on G, we have an open set $G_0 \subset G$ on which f is analytic (in the usual sense with finite values) and so that each point of $G \setminus G_0$ is a pole of f. If we define $f(z) = \infty$ for each $z \in G \setminus G_0$, then $f \colon G \to \hat{\mathbb{C}}$ will be continuous.

To verify continuity at points $z_0 \in G$ where $f(z_0) \in \mathbb{C}$ is finite (so that $z_0 \in G_0$) use $\lim_{z\to z_0} f(z) = f(z_0)$ (or $\lim_{z\to z_0} |f(z) - f(z_0)| = 0$) to deduce

$$\lim_{z\to z_0} \sigma(f(z), f(z_0)) = \lim_{z\to z_0} \frac{2|f(z) - f(z_0)|}{\sqrt{(1 + |f(z)|^2)(1 + |f(z_0)|^2}} = 0.$$

[1]a metric space or topological space where in an open set around each point there is a coordinate system that identifies it with an open piece of $\mathbb{R}^2$; we assume the coordinate functions are continuous from their domain to their range in $\mathbb{R}^2$ and have continuous inverses; usually we also make the slightly technical assumption that there is a countable dense subset in a manifold (or that the topology is second countable if we don't use a metric)

At poles z_0 we know $\lim_{z\to z_0}|f(z)| = \infty$ and so

$$\lim_{z\to z_0}\sigma(f(z), f(z_0)) = \lim_{z\to z_0}\sigma(f(z),\infty) = \lim_{z\to z_0}\frac{2}{\sqrt{1+|f(z)|^2}} = 0$$

Hence $f\colon G\to\hat{\mathbb{C}}$ is continuous.

As $f^{-1}(\mathbb{C})$ is the complement of the poles of f in G, we know that the restriction of f to $f^{-1}(\mathbb{C})$ is analytic. Since $1/f$ is meromorphic on each connected component of G unless $f\equiv 0$ on the component (by Corollary 4.25) we can also see that $1/f$ is analytic on the set $f^{-1}(\hat{\mathbb{C}}\setminus\{0\}$. Hence $f\colon G\to\hat{\mathbb{C}}$ is analytic.

For the converse, if G is connected, $f\colon G\to\hat{\mathbb{C}}$ is analytic but $f\not\equiv\infty$, then the points of $f^{-1}(\infty)$ are isolated in G by Proposition 6.9. Thus f is analytic on $H = f^{-1}(\mathbb{C})$, the points of $G\setminus H = f^{-1}(\infty)$ are isolated in G and at points $z_0\in G\setminus H$

$$0 = \lim_{z\to z_0}\sigma(f(z), f(z_0)) = \lim_{z\to z_0}\sigma(f(z),\infty) = \lim_{z\to z_0}\frac{2}{\sqrt{1+|f(z)|^2}}$$

implies $\lim_{z\to z_0}|f(z)| = \infty$. Thus z_0 is a pole of f by Proposition 4.16.

Remark 6.11 We now extend the notion of analyticity one step further to functions defined on (open subsets of) $\hat{\mathbb{C}}$ (and still allowing values in the extended complex plane).

Definition 6.12 *If $G\subset\hat{\mathbb{C}}$ is open we say that a function $f\colon G\to\hat{\mathbb{C}}$ is* analytic *if*

(i) f is continuous (from G with the metric σ to $\hat{\mathbb{C}}$ with the metric σ)

(ii) f is analytic on $G\cap\mathbb{C}$

(iii) $g(z) = f(1/z)$ is analytic on $\{z\in\mathbb{C} : 1/z\in G\}$ (we include $z=0$ if $\infty\in G$ and $g(0)$ means $f(\infty)$ in that case).

Lemma 6.13 *If $G\subset\mathbb{C}$, the same functions $f\colon G\to\hat{\mathbb{C}}$ are analytic according to Definition 6.8 and Definition 6.12.*

Proof. The type of continuity required in each case is different, as in one case we consider G with the usual metric on $\mathbb{C}$ while in the other we use the metric σ. However, the open subsets $V\subset G$ are the same in either case by Proposition 6.7. Thus looking at continuity in the form $U\subset\hat{\mathbb{C}}$ open $\Rightarrow f^{-1}(U)\subset G$ open, we can see that the continuity requirements are equivalent.

The rest of the proof is straightforward.

Example 6.14 *All rational functions*

$$r(z) = \frac{p(z)}{q(z)}$$

(with p, q polynomials and $q\not\equiv 0$) are analytic $r\colon\hat{\mathbb{C}}\to\hat{\mathbb{C}}$.

Proof. By the Fundamental theorem of algebra, we can factor p and q and cancel common factors to write

$$r(z) = \lambda \frac{\prod_{i=1}^{j}(z-\alpha_i)}{\prod_{i=1}^{k}(z-\beta_i)}$$

where $j \geq 0$, $k \geq 0$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_j\} \cap \{\beta_1, \beta_2, \ldots, \beta_k\} = \emptyset$.

It follows that $r(z)$ is meromorphic on $\mathbb{C}$ with poles at $z = \beta_i$ $(1 \leq i \leq k)$. When suitably interpreted at $z = \beta_i$, $r(z)$ therefore becomes analytic $r \colon \mathbb{C} \to \hat{\mathbb{C}}$. $r(1/z)$ is also rational and so there is a suitable value for $r(1/0) = r(\infty)$ to make $g(z) = r(1/z)$ analytic $g \colon \mathbb{C} \to \hat{\mathbb{C}}$. One can verify that $r \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is continuous.

Remark 6.15 It is in fact the case that rational functions and the constant ∞ are the only analytic functions $f \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. The first step of the proof is to observe that $f^{-1}(\infty)$ cannot be an infinite set if f is not identically ∞.

Proof. As $\hat{\mathbb{C}}$ is compact, an infinite number of distinct points $z_n \in \hat{\mathbb{C}}$ would have to have a convergent subsequence. If the limit was in $\mathbb{C}$, we would have a contradiction to Proposition 6.9. If the limit was ∞, we can switch that to 0 by looking at $f(1/z)$ instead.

For each of points $a \in \mathbb{C}$ with $f(a) = \infty$, we have a pole of the meromorphic function on $\mathbb{C}$. Hence a Laurent series about a with a finite number of negative terms

$$f(z) = \sum_{n=-p}^{\infty} a_n (z-a)^n \quad (0 < |z-a| < \delta)$$

We call the sum of the negative terms the *principal part* of f at a and write it as

$$P_{f,a}(z) = \sum_{n=-p}^{-1} a_n (z-a)^n$$

The sum of the principal parts of f at the (finitely many) points $a \in \mathbb{C}$ where $f(a) = \infty$ produces a rational function $r(z)$. Subtracting from f, gives $f(z) - r(z)$ analytic at all points of $\mathbb{C}$. Hence it has a power series about the origin. As $f(1/z) - r(1/z)$ has a pole or a removable singularity at $z = 0$, it follows (as in Remark 4.18) that $f(z) - r(z) = p(z) =$ a polynomial. Hence $f(z) = r(z) + p(z)$ is rational.

Remark 6.16 *We now describe the version of convergence appropriate for $\hat{\mathbb{C}}$-valued analytic functions.*

Definition 6.17 *For $G \subset \mathbb{C}$ is open we let $C(G, \hat{\mathbb{C}})$ = the set of all continuous functions $: G \to \hat{\mathbb{C}}$, and $H(G, \hat{\mathbb{C}})$ the analytic (or holomorphic) functions. A sequence $(f_n)_{n=1}^{\infty}$ in $C(G, \hat{\mathbb{C}})$ is said to converge* uniformly with respect to σ on compact subsets of G *if for each $K \subset G$ compact we have*

$$\lim_{n\to\infty} \sup_{z \in K} \sigma(f_n(z), f(z)) = 0$$

Lemma 6.18 *If $(f_n)_{n=1}^\infty$ is a sequence in $C(G, \hat{\mathbb{C}})$ that converges uniformly with respect to σ on compact subsets of G to a limit function $f: G \to \hat{\mathbb{C}}$, then $f \in C(G, \hat{\mathbb{C}})$.*

If $(f_n)_{n=1}^\infty$ is a sequence in $H(G, \hat{\mathbb{C}})$ that converges uniformly with respect to σ on compact subsets of G to a limit function $f: G \to \hat{\mathbb{C}}$, then $f \in H(G, \hat{\mathbb{C}})$.

Proof. Omitted. The first part is a standard fact from metric space theory. For the second, analyticity of f at points $z_0 \in G$ where $f(z) \in \mathbb{C}$ is finite can be shown by establishing that for n large f_n has finite values near z_0 and $f_n \to f$ uniformly (with respect to absolute value distance) on some disk about z_0. For points where $f(z_0) = \infty$, we can consider $1/f_n \to 1/f$.

Proposition 6.19 *Let $G \subset \mathbb{C}$ be open. There is a metric $\tilde{\rho}$ on $C(G, \hat{\mathbb{C}})$ so that convergence of sequences in $\tilde{\rho}$ corresponds to uniform convergence with respect to σ on compact subsets of G.*

Proof. Choose an exhaustive sequence $(K_n)_{n=1}^\infty$ of compact subsets of G and define

$$\tilde{\rho}(f, g) = \sum_{n=1}^\infty \frac{1}{2^n} \sup_{z \in K_n} \sigma(f(z), g(z))$$

The rest is not much different from the earlier case where we constructed ρ on $C(G)$. One simplification here results from $\sup_{z \in K_n} \sigma(f(z), g(z)) \leq 2$.

Definition 6.20 *Let $G \subset \mathbb{C}$ be open. A family $\mathcal{F}$ of meromorphic functions on G is called a* normal family *if each sequence $(f_n)_{n=1}^\infty$ in $\mathcal{F}$ has a subsequence that converges uniformly with respect to σ on compact subsets of G to a limit function that is either*

- *a meromorphic f on G, or*
- *the constant function ∞*

Proposition 6.21 *If $G \subset \mathbb{C}$ is connected then a family $\mathcal{F}$ of meromorphic functions on G is a normal family $\iff$ $\mathcal{F}$ is relatively compact when considered as a subset of $(H(G, \hat{\mathbb{C}}), \tilde{\rho})$.*

Proof. In the case of G connected $H(G, \hat{\mathbb{C}}) =$ the meromorphic functions on G together with the constant ∞ (by Theorem 6.10). Proposition 6.19 implies the result.

Definition 6.22 *Let $G \subset \mathbb{C}$ be open. A family $\mathcal{F} \subset C(G, \hat{\mathbb{C}})$ is called equicontinuous (with respect to σ) at a point $z_0 \in G$ if for each $\varepsilon > 0$ it is possible to find $\delta > 0$ so that*

$$|z - z_0| < \delta, f \in \mathcal{F} \Rightarrow \sigma(f(z), f(z_0)) < \varepsilon.$$

Theorem 6.23 (version of Ascoli) *For $G \subset \mathbb{C}$ open, a family $\mathcal{F} \subset C(G, \hat{\mathbb{C}})$ is relatively compact in $(C(G, \hat{\mathbb{C}}), \tilde{\rho})$ if and only if it is equicontinuous at each point of G.*

Proof. Omitted. It is not so different from the Ascoli theorem we had before (5.20), where we left the proof to an appendix. The pointwise boundedness condition is not needed now because $\hat{\mathbb{C}}$ is compact.

Remark 6.24 Montel's theorem does not work for meromorphic functions. There is a theorem for meromorphic functions that takes the place of Montels theorem. It relies on the notion of a *spherical derivative* $\mu(f)$ of a meromorphic function f. At points where $f(z) \in \mathbb{C}$ is finite, the spherical derivative is

$$\mu(f)(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

and the definition at poles (where $f(z) = \infty$) relies on taking reciprocals. When $f(z)$ is neither 0 nor ∞ one can see that

$$\mu\left(\frac{1}{f}\right)(z) = \mu(f)(z)$$

Theorem 6.25 (Marti criterion) *If $G \subset \mathbb{C}$ is open and connected and $\mathcal{F}$ is a family of meromorphic functions on G, then $\mathcal{F}$ is a normal family if and only if it satisfies*

$$\sup_{f \in \mathcal{F}} \sup_{z \in K} \mu(f)(z) < \infty$$

for each compact $K \subset G$.

This means uniform boundedness of the spherical derivative on each compact subset of G.

Proof. Omitted.

Remark 6.26 One application of these ideas (or variants of them) is in the topic of iteration theory for analytic functions.

The most commonly studied case is the situation of rational maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and iteration means considering the composites

$$f, f^2 = f \circ f, f^3 = f \circ f^2, \ldots$$

Slightly more precisely, one considers a point $z \in \hat{\mathbb{C}}$ and the long term behaviour of the iterated images

$$z, f(z), f^2(z) = f(f(z)), f^3(z), \ldots$$

and examines whether there is a trend.

The 'well-behaved set' of z (for a given rational f) is called the Fatou set $F(f)$ of f. It is the largest open subset of $\hat{\mathbb{C}}$ on which the family of iterates $\{f, f^2, f^3, \ldots\}$ is an equicontinuous family. This means that a small change in z has a small long term effect on the iterates $f^n(z)$.

The Julia set $J(f)$ is the complement in $\hat{\mathbb{C}}$ of the Fatou set. The Julia set is the 'bad' set where a small change in z can produce a major change in the behaviour of the sequence $f^n(z)$ of iterates.

One basic theorem is that for polynomials f of degree 2 or more, ∞ is in the Fatou set and the Julia set is not empty. The Mandelbrot set consists of those values of a parameter $c \in \mathbb{C}$ for which the the Julia set $J(p_c)$ is connected, where $p_c(z) = z^2 + c$.

For lack of time, we won't deal with this topic.