Chapter 5: The metric space $H(G)$

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Remark 5.1 For $G \subset \mathbb{C}$ open, we use the notation

$$H(G) = \{f : G \to \mathbb{C} \text{ analytic}\}$$

for the set (space) of all holomorphic functions on G and

$$C(G) = \{f : G \to \mathbb{C} \text{ continuous}\}$$

for the set (space) of all continuous functions on G.

The main change of perspective now is that we move from looking at functions one at a time to looking at properties of the whole space of functions.

It is a more simple fact that $H(G)$ and $C(G)$ are algebras than the fact that $M(G)$ is an algebra. We define vector space operations on $f, g \in C(G)$ and $\lambda \in \mathbb{C}$ via

$$(f + g)(z) = f(z) + g(z) \text{ and } (\lambda f)(z) = \lambda f(z)$$

and a multiplication on $C(G)$ by $(fg)(z) = f(z)g(z)$. This makes $C(G)$ a commutative algebra over $\mathbb{C}$ and $H(G)$ a subalgebra.

We are interested in what it might mean for a sequence of functions $(f_n)_{n=1}^{\infty}$ (in $H(G)$ say) to converge to a limit $f \in H(G)$. Most of the ideas are the same initially for sequences in $C(G)$ and but we will come later to properties special to holomorphic functions.

The simplest way to define $\lim_{n\to\infty} f_n = f$ is the notion of *pointwise convergence* on G. This means that for each $z \in G$ we have $\lim_{n\to\infty} f_n(z) = f(z)$ (convergence as sequences of complex numbers). Though it easy to define, it is a kind of convergence that has rather unpleasant properties. Perhaps it is more accurate to say that it fails to have many properties we would like to have linking behaviour of the functions f_n to the behaviour of the limit function f. These drawbacks include

(i) We can define pointwise convergence for sequences in $H(G)$ without insisting that the limit function be in $H(G)$ or even $C(G)$. Then there are sequences of holomorphic functions that converge pointwise to discontinuous limits f. Such examples are hard to come by and the easiest way to show they can be found is to invoke Runges theorem (something we will come to later).

(ii) We can exhibit a sequence in $C(\mathbb{C})$ that converges to a discontinuous limit. For example $f_n(z) = e^{-n|z|}$ has

$$\lim_{n\to\infty} f_n(z) = \begin{cases} 1 & z = 0 \\ 0 & z \neq 0 \end{cases}$$

(iii) There is no distance on $C(G)$ or $H(G)$ so that $f_n \to f$ pointwise on G is equivalent to convergence in distance, that is to

$$\lim_{n\to\infty} \text{distance}\,(f_n, f) = 0$$

This means in fact that the behaviour of pointwise convergence in $H(G)$ or $C(G)$ belongs in a more complicated theory than the theory of metric spaces. In metric spaces many, but not all, familiar properties of limits of sequences of scalars are still true.

The next simplest notion to consider is *uniform convergence on* G. Recall that $f_n \to f$ uniformly on G (as $n \to \infty$) means the following

Given $\varepsilon > 0$ there exists N so that

$$n > N, z \in G \Rightarrow |f_n(z) - f(z)| < \varepsilon$$

With a small bit of work, we could check that this is equivalent to the following formulation

Given $\varepsilon > 0$ there exists N so that

$$n > N \Rightarrow \sup_{z\in G} |f_n(z) - f(z)| < \varepsilon$$

In this way we can see a definition of distance we can use to describe this kind of convergence

$$\text{distance}\,(g, h) = d_G(g, h) = \sup_{z\in G} |g(z) - h(z)|$$

and this looks promising. We do know that $f_n \to f$ uniformly on G and each $f_n \in C(G)$ implies $f \in C(G)$ ('uniform limits of continuous functions are continuous'). We also know a similar fact about sequences in $H(G)$ (see Exercises 2 where we saw that Moreras theorem could be used to show that uniform limits of sequences of holomorphic functions are holomorphic).

However, there are still drawbacks:

(a) If we apply this convergence in $H(\mathbb{C})$ (that is to entire functions) we find that $d_{\mathbb{C}}(f_n, f) < \varepsilon < \infty$ implies that $f_n(z) - f(z)$ is a bounded entire function, hence a constant by Liouville's theorem. So in $H(\mathbb{C})$ the sequences of functions that converge uniformly are all of the form $f_n(z) = f(z) + c_n$ with c_n constants converging to 0 (except for some finite number of n where f_n can be anything). Thus a very restrictive notion of convergence.

(b) This is linked to the fact that $d_G(g,h)$ need to be finite for $f, g \in C(G)$ (or even $f, g \in H(G)$). So our notion of 'distance' is not so great as it is not always defined.

It is a good notion to use if we looked only at bounded continuous functions on G (or bounded holomorphic functions, but we see that if $G = \mathbb{C}$ then there are only constant bounded holomorphic functions). We will not look into these spaces of bounded functions, though they have been studied extensively, as they are a rather more advanced topic.

(c) The notion of uniform convergence does not cover all the cases where we have already encountered limits of sequences of functions, especially power series. For example we have

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = \lim_{n\to\infty} \sum_{j=1}^{n} z^j = \lim_{n\to\infty} s_n(z) \quad (|z| < 1)$$

but we don't have uniform convergence for $z \in D(0,1)$. Similarly we have

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{z^n}{n!} = \lim_{n\to\infty} s_n(z) \quad (z \in \mathbb{C})$$

but no unform convergence on $\mathbb{C}$. In general for a power series with radius of convergence $R > 0$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n = \lim_{n\to\infty} \sum_{j=1}^{n} a_j (z-a)^j = \lim_{n\to\infty} s_n(z) \quad (|z-a| < R)$$

we do not have unform convergence for $z \in D(a,R)$ but we do have uniform convergence for $|z-a| \leq r$ with any $r < R$. (This follows from the Weierstrass M-test.)

This last point shows the way to proceed in general.

Definition 5.2 *If $(f_n)_{n=1}^{\infty}$ is a sequence of functions $f_n\colon G \to \mathbb{C}$ defined on an open $G \subset \mathbb{C}$ and $f\colon G \to \mathbb{C}$ is another function, then we say the sequence converges to f* uniformly on compact subsets of G *if the following is satisfied*

> *for each compact subset $K \subset G$ the sequence of restrictions $(f_n \mid_K)_{n=1}^{\infty}$ converges uniformly on K to $f \mid_K$.*

Spelling this out more it says

> for each compact subset $K \subset G$ and any given $\varepsilon > 0$ we can find N so that
>
> $$n > N, z \in K \Rightarrow |f_n(z) - f(z)| < \varepsilon$$

This can also be reformulated using a distance on each K, or on $C(K) = \{g\colon K \to \mathbb{C} \text{ continuous}\}$, defined by

$$d_K(g,h) = \sup_{z\in K} |g(z) - h(z)|$$

For $g, h \in C(K)$ this supremum is always finite (continuous complex-valued functions on compact sets are always bounded and there is even some $z \in K$ where $|g(z) - h(z)|$ achieves its maximum). This d_K distance makes $C(K)$ a *metric space*, that is d_K satisfies the following properties that seem to be natural for familiar distances:

(i) $d_K(g, h) \geq 0 \quad \forall g, h \in C(K)$

(ii) $d_K(g, h) = d_K(h, g) \quad \forall g, h \in C(K)$

(iii) (triangle inequality)

$$d_K(g_1, g_3) \leq d_K(g_1, g_2) + d_K(g_2, g_3) \quad (\forall g_1, g_2, g_3 \in C(K))$$

[These properties so far make $C(K)$ with d_K a pseudo-metric space (or sometimes called a semi-metric space).]

(iv) $g, h \in C(K)$ and $d_K(g, h) = 0 \Rightarrow g = h$.

We can say $f_n \to f$ uniformly on compact subsets if and only if the following is true

for each compact $K \subset G$, $\lim_{n\to\infty} d_K(f_n, f) = 0$

What we have now is infinitely many distances d_K to consider in order to describe uniform convergence on compact sets, but we will show soon how to manage with just one distance. Meantime we show that this kind of convergence has desirable properties (and that it covers the power series situation).

Proposition 5.3 *Let $G \subset \mathbb{C}$ be open and $(f_n)_{n=1}^\infty$ a sequence in $H(G)$ that converges uniformly on compact subsets of G to some limit function $f: G \to \mathbb{C}$. Then $f \in H(G)$.*

Proof. Analyticity of the limit is a local property, something to be shown at each point $a \in G$. We need to show that $f'(a)$ exists.

For this fix $a \in G$ and choose $r > 0$ so that $D(a, r) \subset G$. Then pick δ with $0 < \delta < r$ so that $K = \bar{D}(a, \delta) \subset G$ is a compact subset. So $f_n \to f$ uniformly on K and so $f_n \to f$ uniformly on $D(a, \delta)$. By Exercises 2 question 3 (the one that used Moreras theorem) we have that the restriction of f to $D(a, \delta)$ is analytic. So $f'(a)$ exists.

Proposition 5.4 *If $G \subset \mathbb{C}$ is open, then we can find a sequence $(K_n)_{n=1}^\infty$ of compact subsets of G such that*

(i) $\bigcup_{n=1}^\infty K_n = G$

(ii) $K_n \subset (K_{n+1})^\circ =$ *the interior of* K_{n+1} *for each* $n = 1, 2, \ldots$

(iii) if $K \subset G$ is any compact subset, then there is some n with $K \subset K_n$

We call such a sequence an exhaustive sequence of compact subsets *of G.*

Proof. We already showed how to construct such a sequence in the proof of 4.21 (and see 4.22). When $G = \mathbb{C}$ it was easy — just put $K_n = \bar{D}(0, n)$ and for $G \neq \mathbb{C}$ we took

$$K_n = \{z \in G : \text{dist}(z, \mathbb{C} \setminus G) \geq \frac{1}{n} \text{ and } |z| \leq n\}.$$

Recall also

$$\mathbb{C} \setminus K_n = \{w \in \mathbb{C} : |w| > n\} \cup \bigcup_{w \in \mathbb{C} \setminus G} D\left(w, \frac{1}{n}\right)$$

and $z \in K_n \Rightarrow D\left(z, \frac{1}{n} - \frac{1}{n+1}\right) \subset K_{n+1}$.

We did not note then that the K_n satisfy (iii) but if $K \subset G$ is compact then $K \subset \bigcup_{n=1}^{\infty} K_n \subset \bigcup_{n=1}^{\infty} (K_{n+1})^\circ$ and so we have an open cover of K by the sets $(K_{n+1})^\circ$. Hence there is a finite subcover $K \subset \bigcup_{n=1}^{N} (K_{n+1})^\circ = (K_{N+1})^\circ \subset K_{N+1}$. This shows (iii) holds.

Lemma 5.5 *Suppose $G \subset \mathbb{C}$ is open and (K_n) is an exhaustive sequence of compact subsets of G. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions $f_j \in C(G)$ and $f \in C(G)$. Then $f_j \to f$ uniformly on compact subsets of G (as $j \to \infty$) if and only if*

$$\forall n = 1, 2, \ldots, \quad \lim_{j \to \infty} d_{K_n}(f_j, f) = 0$$

This means that instead of considering the vastly infinite (uncountably infinite) number of compact $K \subset G$ it is enough to look only at K in the sequence K_n.

Proof. Since each $K_n \subset G$ is a compact subset it is clear that if $f_j \to f$ uniformly on compact subsets of G then $\lim_{j \to \infty} d_{K_n}(f_j, f) = 0$ for each n.

Conversely, suppose we know $\lim_{j \to \infty} d_{K_n}(f_j, f) = 0$ for each n and we take any $K \subset G$ compact. Then there is some n with $K \subset K_n$ and it is fairly clear then that

$$d_K(f_j, f) \leq d_{K_n}(f_j, f) \to 0 \text{ as } j \to \infty$$

Example 5.6 It is now clear that in the power series examples with radius of convergence $R > 0$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n = \lim_{n \to \infty} \sum_{j=1}^{n} a_j (z-a)^j = \lim_{n \to \infty} s_n(z) \quad (|z-a| < R)$$

we do have $s_n \to f$ uniformly on compact subsets of $D(a, R)$.

The reason is every compact subset $K \subset D(a, R)$ is contained in a subdisc $D(a, r)$ with $0 < r < R$ (where we do have uniform convergence by the Weierstrass M-test argument). Another way to see this is to work out what the standard construction gives for K_n when $G = D(a, R)$ (for n large, a sequence of closed discs of radii $< R$ centered at a).

Remark 5.7 These d_K (or d_{K_n}) have a nice property that they are associated with a *semi-norm* on $C(G)$. A semi-norm is a way of associating a length to elements of a vector space that has all the usual properties of lengths of vectors except non-zero elements are allowed to have length $= 0$. (Also we are not saying that there is any inner product around.)

If we define for $K \subset G$ compact (here $G \subset \mathbb{C}$ is open as usual) and $f \in C(G)$

$$\|f\|_K = \sup_{z \in K} |f(z)|$$

then we have the following characteristic semi-norm properties satisfied

(i) $\|f\|_K \geq 0$ for each $f \in C(G)$

(ii) (triangle inequality) $\|f + g\|_K \leq \|f\|_K + \|g\|_K$ holds for all $f, g \in C(G)$

(iii) $\lambda \in \mathbb{C}, f \in C(G) \Rightarrow \|\lambda f\|_K = |\lambda| \|f\|_K$

The link between the semi-norm $\| \cdot \|_K$ and the pseudo-metric d_K is

$$d_K(g, h) = \|g - h\|_K$$

and all semi-norms give rise to a pseudo-metric in this way.

We will now show that there is a single metric on $C(G)$ that describes uniform convergence on compact subsets, but we will lose the connection with a norm (or semi-norm).

Lemma 5.8 *Let $G \subset \mathbb{C}$ be open and $K \subset G$ compact. Define $\rho_K(f, g)$ for $f, g \in C(G)$ by*

$$\rho_K(f, g) = \frac{d_K(f, g)}{1 + d_K(f, g)}$$

Then

(i) ρ_K is a pseudo metric on $C(G)$

(ii) $0 \leq \rho_K(f, g) < 1$ for all $f, g \in C(G)$

(iii) If $(f_n)_{n=1}^\infty$ is a sequence of functions in $C(G)$ and $f \in C(G)$, then

$$\lim_{n \to \infty} d_K(f_n, f) = 0 \iff \lim_{n \to \infty} \rho_K(f_n, f) = 0$$

Proof.

(i) This is the main part requiring proof. It is clear that $\rho_K(f, g) = \rho_K(g, h) \geq 0$ and the part that requires an argument is that ρ_K satisfies the triangle inequality.

Let $\rho(t) = t/(1+t)$ for $t \geq 0$ and note that $\rho'(t) = 1/(1+t)^2 \geq 0$ so that $\rho(t)$ is increasing (for $t \geq 0$). Now the triangle inequality holds for d_K and so we can say

$$\begin{aligned}
\rho_K(f_1, f_3) &= \rho(d_K(f_1, f_3)) \\
&\leq \rho(d_K(f_1, f_2) + d_K(f_2, f_3)) \\
&= \frac{d_K(f_1, f_2) + d_K(f_2, f_3)}{1 + d_K(f_1, f_2) + d_K(f_2, f_3)} \\
&= \frac{d_K(f_1, f_2)}{1 + d_K(f_1, f_2) + d_K(f_2, f_3)} + \frac{d_K(f_2, f_3)}{1 + d_K(f_1, f_2) + d_K(f_2, f_3)} \\
&\leq \frac{d_K(f_1, f_2)}{1 + d_K(f_1, f_2)} + \frac{d_K(f_2, f_3)}{1 + d_K(f_2, f_3)} \\
&= \rho_K(f_1, f_2) + \rho_k(f_2, f_3)
\end{aligned}$$

(ii) This is clear from the definition of ρ_K

(iii) If $\lim_{n\to\infty} d_K(f_n, f) = 0$ then $\lim_{n\to\infty} \rho_K(f_n, f) = 0$ because the function $\rho(t) = t/(1+t)$ is continuous at $t = 0$ and $\rho_K(f_n, f) = \rho(d_K(f_n, f))$.

Conversely if $\lim_{n\to\infty} \rho_K(f_n, f) = 0$, then $\lim_{n\to\infty} d_K(f_n, f) = 0$ because $d_K(f_n, f) = \rho_K(f_n, f)/(1 - \rho_K(f_n, f))$.

Proposition 5.9 *Let $G \subset \mathbb{C}$ be an open subset and $(K_n)_{n=1}^\infty$ an exhaustive sequence of compact subsets of G. Then*

$$\rho(f, g) = \sum_{n=1}^\infty \frac{1}{2^n} \rho_{K_n}(f, g) \qquad (f, g \in C(G))$$

defines a metric on $C(G)$. If $(f_j)_{j=1}^\infty$ is a sequence in $C(G)$ and $f \in C(G)$ then

$$\lim_{j\to\infty} \rho(f_j, f) = 0$$

if and only if $f_j \to f$ uniformly on compact subsets of G (as $j \to \infty$).

In other words this metric ρ is such that convergence in this distance corresponds to uniform convergence on compact subsets.

Proof. To show that ρ is a pseudo metric on $C(G)$ is not difficult using Lemma 5.8. The series defining $\rho(f, g)$ is convergent since it is smaller than $\sum_{n=1}^\infty 1/2^n = 1$ and it clearly has a non-negative sum. The triangle inequality follows by applying Lemma 5.8 as does $\rho(f, g) = \rho(g, f)$. Finally $\rho(f, g) = 0 \Rightarrow \rho_{K_n}(f, g) = 0 \forall n \Rightarrow d_{K_n}(f, g) = 0 \forall n \Rightarrow f(z) = g(z) \forall z \in \bigcup_{n=1}^\infty K_n = G \Rightarrow f = g$

If $f_j \to f$ uniformly on compact subsets of G, then $\lim_{j\to\infty} d_{K_n}(f_j, f) = 0$ for each n and hence $\lim_{j\to\infty} \rho_{K_n}(f_j, f) = 0$ (for each n). To show that $\lim_{j\to\infty} \rho(f_j, f) = 0$, start with $\varepsilon > 0$ given and choose N large enough that

$$\sum_{n=N+1}^\infty \frac{1}{2^n} < \frac{\varepsilon}{2}$$

Then we have

$$\rho(f_j, f) = \sum_{n=1}^{\infty} \frac{1}{2^n}\rho_{K_n}(f_j, f) \leq \sum_{n=1}^{N} \frac{1}{2^n}\rho_{K_n}(f_j, f) + \sum_{n=N+1}^{\infty} \frac{1}{2^n} < \sum_{n=1}^{N} \frac{1}{2^n}\rho_{K_n}(f_j, f) + \frac{\varepsilon}{2}$$

Since $\lim_{j\to\infty} \sum_{n=1}^{N} \frac{1}{2^n}\rho_{K_n}(f_j, f) = 0$, if j is large enough (say $j > j_0$) then this finite sum is $< \varepsilon/2$ and $\rho(f_j, f) < \varepsilon$.

Conversely, if $\lim_{j\to\infty} \rho(f_j, f) = 0$ then notice that, for any fixed n,

$$\rho_{K_n}(f_j, f) \leq 2^n \sum_{m=1}^{\infty} \frac{1}{2^m}\rho_{K_m}(f_j, f) = 2^n\rho(f_j, f) \to 0 \text{ as } j \to \infty$$

So $\lim_{j\to\infty} \rho_{K_n}(f_j, f) = 0$ for each n, hence by Lemma 5.8(iii) $\lim_{j\to\infty} d_{K_n}(f_j, f) = 0$ and by Lemma 5.5, $f_j \to f$ uniformly on compact subsets of G as $j \to \infty$.

Remark 5.10 From now on we will frequently consider $H(G)$ (or $C(G)$) as equipped with a metric ρ of the above type. So $(H(G), \rho)$ is now a metric space.

We can define open sets in a metric space by analogy with the way we define open subsets in $\mathbb{C}$. Sometimes it may help to consider a subset of $H(G)$ as a family of analytic functions (which means the same as a set of functions). If we fix $f_0 \in H(G)$ and $r > 0$ we can define the *ball about f_0 of radius r* as

$$B_\rho(f_0, r) = \{f \in H(G) : \rho(f, f_0) < r\}$$

and then we can define a subset $\mathcal{F} \subset H(G)$ to be open if $f_0 \in \mathcal{F} \Rightarrow \exists r > 0$ with $B_\rho(f_0, r) \subset \mathcal{F}$.

We can define compactness for subsets $\mathcal{F} \subset H(G)$ by requiring that every open cover has a finite subcover. But in metric spaces, we always have an alternative way to describe compactness via sequences (as we do in $\mathbb{C}$). A family $\mathcal{F} \subset H(G)$ is compact if and only if every sequence $(f_n)_{n=1}^{\infty}$ of functions $f_n \in \mathcal{F}$ has a subsequence $(f_{n_j})_{j=1}^{\infty}$ with a limit $f \in \mathcal{F}$. (So $\lim_{j\to\infty} \rho(f_{n_j}, f) = 0$.)

Lemma 5.11 *For $G \subset \mathbb{C}$ open and let $\mathcal{F} \subset H(G)$ be a family of holomorphic functions (or subset of $H(G)$). Consider two metrics ρ_1, ρ_2 on $H(G)$ constructed as above (from exhaustive sequences of compact subsets of G). Then $\mathcal{F}$ is compact in $(H(G), \rho_1)$ if and only if it is compact in the metric space $(H(G), \rho_2)$.*

Proof. As noted above compactness of subsets of a metric space can be characterised using limits of sequences. But both metrics ρ_1 and ρ_2 have the same convergent sequences by Proposition 5.9.

Remark 5.12 Closures and closed subsets of $(H(G), \rho)$ can also be characterised using limits of sequences. Hence they are also the same for different metrics ρ constructed as before.

Open subsets of $(H(G), \rho)$ are also the same in different such metrics ρ (since open sets are complements of closed sets).

Similar remarks apply to $(C(G), \rho)$.

Proposition 5.13 *Let $G \subset \mathbb{C}$ be open. Then $H(G)$ is a closed subset of $C(G)$ (when we use a metric of the type constructed above on $C(G)$).*

(As $H(G)$ is a vector subspace of $C(G)$, it is commonly called a closed subspace.)

Proof. Convergent sequences of functions f_n in $H(G)$ with $\lim_{n\to\infty} f_n = f \in C(G)$ have limits $f \in H(G)$ by Proposition 5.3. It follows that $H(G)$ is closed.

Lemma 5.14 *Let $G \subset \mathbb{C}$ be open and ρ a metric on $C(G)$ constructed as before. Let $K \subset G$ be compact. Then the map*

$$f \mapsto \|f\|_K \colon C(G) \to \mathbb{R}$$

is continuous.

Proof. Continuity on metric spaces can be described via sequences. It suffices to prove that $\lim_{n\to\infty} \rho(f_n, f) = 0 \Rightarrow \lim_{n\to\infty} \|f_n\|_K = \|f\|_K$.

This follows because a version of the triangle inequality says

$$|\|f_n\|_K - \|f\|_K| \leq \|f - f_n\|_K \to 0$$

(as $n \to \infty$) by uniform convergence of the sequence $(f_n)_{n=1}^\infty$ to f on K.

Definition 5.15 *Let $G \subset \mathbb{C}$ be open. A subset $\mathcal{F}$ of $C(G)$ (or $H(G)$) is called* relatively compact *if and only if its closure is compact. (The closure will be the set of all possible limits of convergent sequences $(f_n)_{n=1}^\infty$ with each $f_n \in \mathcal{F}$.)*

A subset $\mathcal{F}$ is called bounded *if for each $K \subset G$ compact*

$$\sup_{f\in\mathcal{F}} \|f\|_K < \infty$$

(This could be phrased as 'uniformly bounded on compact subsets', but the shorter term 'bounded' is usually used. In a way, this is the only sensible notion of 'bounded'. Defining bounded in terms of the distance ρ is not much use as all distances are bounded by 1.)

Lemma 5.16 *Let $G \subset \mathbb{C}$ be open and $\mathcal{F} \subset C(G)$ (or $\mathcal{F} \subset H(G)$) a subset.*

(i) $\mathcal{F}$ is relatively compact if and only if every sequence $(f_n)_{n=1}^\infty$ in $\mathcal{F}$ has a convergent subsequence $(f_{n_j})_{j=1}^\infty$ (convergent to some limit in $C(G)$)

(ii) Relatively compact families $\mathcal{F}$ are bounded.

Proof.

(i) (This is actually a general fact, true in any metric space.) If the closure of $\mathcal{F}$ is compact, then a sequence $(f_n)_{n=1}^\infty$ in $\mathcal{F}$ is also a sequence in the closure. As the closure is compact there is a subsequence $(f_{n_j})_{j=1}^\infty$ converging to a limit (in the closure, hence in $C(G)$).

Going the other way, suppose we know every sequence in $\mathcal{F}$ has a convergent subsequence. Take a sequence $(g_n)_{n=1}^\infty$ in the closure $\bar{\mathcal{F}}$. Then, for each n, we can find $f_n \in \mathcal{F}$ so that

$\rho(g_n, f_n) < 1/n$. By assumption, there is a subsequence $(f_{n_j})_{j=1}^\infty$ which converges to some $f \in C(G)$. But then $g_{n_j} \to f$ as $j \to \infty$ because

$$\rho(g_{n_j}, f) \leq \rho(g_{n_j}, f_{n_j}) + \rho(f_{n_j}, f) \leq \frac{1}{n_j} + \rho(f_{n_j}, f) \to 0 \quad (\text{as } j \to \infty).$$

(ii) Let $\mathcal{F} \subset C(G)$ be relatively compact. If it is not bounded, then there is some $K \subset G$ compact so that

$$\sup_{f \in \mathcal{F}} \|f\|_K = \infty.$$

So for each n, we can find $f_n \in \mathcal{F}$ with $\|f_n\|_K > n$. A convergent subsequence $(f_{n_j})_{j=1}^\infty$ exists (with a limit $f \in C(G)$). By Lemma 5.14,

$$\lim_{j \to \infty} \|f_{n_j}\|_K = \|f\|_K$$

but that contradicts $\|f_{n_j}\|_K > n_j \to \infty$ as $j \to \infty$.

Theorem 5.17 (Montels Theorem) *Let $G \subset \mathbb{C}$ be open and $\mathcal{F} \subset H(G)$ a family of analytic functions. Then $\mathcal{F}$ is relatively compact if and only if $\mathcal{F}$ is bounded (in the sense of Definition 5.15, that is uniformly on compact subsets).*

Remark 5.18 Though this theorem corresponds exactly with the situation for subsets of $\mathbb{C}$ and finite-dimensional vector spaces like $\mathbb{R}^n$, a similar result is not usually true in infinite dimensional spaces.

The same statement fails in $C(G)$. An example to show that is $\mathcal{F} = \{f_n : n = 1, 2, \ldots\}$, $f_n(z) = \exp(-n|z|)$ and G any open set containing the origin. The family is bounded because $|f_n(z)| = f_n(z) \leq 1$, but there is no subsequence of the sequence $(f_n)_{n=1}^\infty$ that converges in $C(G)$. If a subsequence did converge in $(C(G), \rho)$ to some limit $f \in C(G)$, then the subsequence would have to converge pointwise to the same limit function. This forces $f(0) = 1$ and $f(z) = 0$ for $z \neq 0$. So f cannot be in $C(G)$.

Our proof of Montels Theorem will require the corresponding theorem for families in $C(G)$. That theorem (the Arzela-Ascoli Theorem) is more complicated to state and also a bit long to prove. We will relegate the proof of it to an appendix.

Definition 5.19 *Let $G \subset \mathbb{C}$ be open $z_0 \in G$ a point and $\mathcal{F} \subset C(G)$ a family of continuous functions on G. Then the family $\mathcal{F}$ is called* equicontinuous *at z_0 if for each $\varepsilon > 0$ there is some $\delta > 0$ so that*

$$|z - z_0| < \delta, f \in \mathcal{F} \Rightarrow |f(z) - f(z_0)| < \varepsilon.$$

(Perhaps it is worth comparing this to uniform continuity of a single function on a set E. There f is fixed and z, z_0 are any two points of the set E with $|z - z_0| < \delta$. Here, the δ required is as in the condition for continuity of a single function f at z_0, but the same δ has to work for all $f \in \mathcal{F}$.)

Theorem 5.20 (Arzela-Ascoli Theorem) *Let $G \subset \mathbb{C}$ be open and $\mathcal{F} \subset C(G)$ a family of continuous functions on G. Then $\mathcal{F}$ is relatively compact in $(C(G), \rho)$ if and only if it satisfies both of the following conditions:*

(i) $\mathcal{F}$ is pointwise bounded on G (that is, for each point $z_0 \in G$, we have $\sup_{f \in \mathcal{F}} |f(z_0)| < \infty$)

(ii) $\mathcal{F}$ is equicontinuous at each point of G.

Proof. (of Theorem 5.17 using Theorem 5.20) One direction is already covered by Lemma 5.16 (i). If $\mathcal{F} \subset H(G)$ is relatively compact then it must be bounded.

For the other direction, we use Theorem 5.20. Assume $\mathcal{F} \subset H(G)$ is bounded.

Then $\mathcal{F}$ is certainly pointwise bounded since points $z_0 \in G$ make singleton compact sets $K = \{z_0\}$ (and $\mathcal{F}$ is uniformly bounded on K).

Next we claim $\mathcal{F}$ is equicontinuous at each $z_0 \in G$. Fix $z_0 \in G$ and $\varepsilon > 0$.

We can find $r > 0$ with $D(z_0, r) \subset G$. If $0 < \delta_0 < r/2$, then the closed disk $K = \bar{D}(z_0, 2\delta_0) \subset D(z_0, r) \subset G$ is a compact subset of G. So

$$M = \sup_{f \in \mathcal{F}} \|f\|_K < \infty.$$

By the Cauchy integral formula, for $z \in D(z_0, \delta_0)$ we have

$$f'(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = 2\delta_0} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$$

and so we can estimate

$$|f'(z)| \leq \frac{1}{2\pi}(2\pi(2\delta_0))\frac{M}{\delta_0^2} = \frac{2M}{\delta_0} = M_1 \quad \text{(say)}$$

(using $|\zeta - z| = |(\zeta - z_0) - (z - z_0)| \geq |\zeta - z_0| - |z - z_0| = 2\delta_0 - |z - z_0| \geq 2\delta_0 - \delta_0 = \delta_0$). Thus for $|z - z_0| < \delta_0$ we have

$$|f(z) - f(z_0)| = \left| \int_{z_0}^{z} f'(\zeta)\, d\zeta \right| \leq |z - z_0| M_1$$

Thus if we take $\delta = \min(\delta_0, \varepsilon/M_1)$ (which is independent of $f \in \mathcal{F}$), we have

$$|z - z_0| < \delta, f \in \mathcal{F} \Rightarrow |f(z) - f(z_0)| < \varepsilon$$

Thus, by the Ascoli theorem 5.20, we know that $\mathcal{F}$ is relatively compact if we view it as a family of continuous functions ($\mathcal{F} \subset H(G) \subset C(G)$). In other words its closure in $C(G)$ is compact. But, since $H(G)$ is closed in $C(G)$, the closure of $\mathcal{F}$ in $C(G)$ is actually contained in $H(G)$ and so is the same as its closure in $H(G)$. Hence $\mathcal{F}$ is relatively compact in $H(G)$.

Theorem 5.21 *Suppose $G \subset \mathbb{C}$ is open and $\mathcal{F} \subset H(G)$ a bounded subset. If a sequence $(f_n)_{n=1}^\infty$ in $\mathcal{F}$ is pointwise convergent to a function $f\colon G \to \mathbb{C}$, then the sequence is automatically uniformly convergent on compact subsets (and the limit $f \in H(G)$).*

(In other words, for *bounded* sequences of analytic functions, pointwise convergence is after all the same as uniform convergence on bounded sets. It follows that if a sequence of analytic functions $f_n\colon G \to \mathbb{C}$ converges pointwise to a limit $f\colon G \to \mathbb{C}$ and if f fails to be differentiable or continuous at a point $z_0 \in G$, then the sequence cannot be bounded (uniformly on compact subsets of G). In fact we can replace G be a small disc $D(z_0, \delta) \subset G$ and say that the sequence could not be bounded on any such disc. So

$$\sup_n (\sup\{|f_n(z)| : |z - z_0| < \delta\}) = \infty$$

for each $\delta > 0$ (small enough that $D(z_0, \delta) \subset G$). This makes it quite hard to find such sequences f_n. As stated before, we can use Runges theorem to show there are such sequences, once we find out about Runges theorem.)

Proof. By Montels Theorem 5.17, the sequence has a convergent subsequence $(f_{n_j})_{j=1}^\infty$. So the subsequence converges in $(H(G), \rho)$ (or equivalently, uniformly on compact subsets of G) to some limit $g \in H(G)$. As singleton subsets $K = \{z\} \subset G$ are compact it follows that the subsequence converges pointwise to g. That is

$$g(z) = \lim_{j\to\infty} f_{n_j}(z) = \lim_{n\to\infty} f_n(z) = f(z) \quad \text{(each } z \in G)$$

and so $f = g \in H(G)$.

To show that the sequence $(f_n)_{n=1}^\infty$ converges, uniformly on compact subsets of G, to f we need to know that for each $K \subset G$ compact

$$\lim_{n\to\infty} \|f_n - f\|_K = 0.$$

If that fails to be so, it fails for some $K \subset G$ compact. Failing for K means we can find $\varepsilon > 0$ so that $\|f_n - f\|_K \geq \varepsilon$ for infinitely many n. That means a subsequence $(f_{n_k})_{k=1}^\infty$ where $\|f_{n_k} - f\|_K \geq \varepsilon$ for all k.

Using Montels theorem, we can find a subsequence $(f_{n_{k_j}})_{j=1}^\infty$ of the subsequence which converges in $(H(G), \rho)$. We can rename this sub-subsequence as $(f_{m_j})_{j=1}^\infty$. Repeating the argument at the beginning of the proof we can see that this subsequence must converge to f and so

$$\lim_{j\to\infty} \|f_{m_j} - f\|_K = 0$$

contradicting

$$\|f_{m_j} - f\|_K = \|f_{n_{k_j}} - f\|_K \geq \varepsilon > 0 \quad (\forall j).$$

Thus we must have $f_n \to f$ in $(H(G), \rho)$.

Theorem 5.22 (Osgoods theorem) *Let $G \subset \mathbb{C}$ be open and $(f_n)_{n=1}^\infty$ a sequence of analytic functions $f_n \in H(G)$ that converges pointwise to a limit function $f \colon G \to \mathbb{C}$.*

Then there is a dense open subset $G_0 \subset G$ so that the restriction of f to G_0 is analytic and the restriction of the sequence $(f_n)_{n=1}^\infty$ converges uniformly on compact subsets of G_0 to $f \mid_{G_0}$.

The proof of this relies on the previous theorem (5.21) and the Baire category theorem.

The idea is to take

$$S_m = \{z \in G : \sup_n |f_n(z)| \leq m\} = \bigcap_{n=1}^\infty \{z \in G : |f_n(z)| \leq m\}$$

$G_m = (S_m)^\circ$ the interior of S_m and $G_0 = \bigcup_{m=1}^\infty G_m$.

Clearly G_0 is open. The Baire Category theorem can be used to show that G_m is not empty for m big enough, and in fact that G_0 is dense in G. This is the main part of the proof.

Once these facts are established, we can see that every compact subset $K \subset G_0 = \bigcup_{m=1}^\infty G_m$ and so the G_m form an open cover of K. Hence K is contained in a finite union $\bigcup_{m=1}^{m_0} G_m = G_{m_0}$ and so the sequence f_n is uniformly bounded on K (by m_0).

From Theorem 5.21, the rest of the result follows.

We will leave the rest of the details to an appendix.

A Proof of Arzela-Ascoli Theorem

First, the 'easy' direction of the theorem.

Proposition A.1 *Let $G \subset \mathbb{C}$ be open and $\mathcal{F} \subset C(G)$ a family of continuous functions on G. Suppose that $\mathcal{F}$ is a relatively compact in $(C(G), \rho)$. Then it satisfies both of the following conditions:*

(i) $\mathcal{F}$ is pointwise bounded on G

(ii) $\mathcal{F}$ is equicontinuous at each point of G.

Proof. We know from Lemma 5.16 (i) that $\mathcal{F}$ is bounded (uniformly on compact subsets) and so it is pointwise bounded (because singleton subsets $K = \{z_0\} \subset G$ are compact).

To show it must be equicontinuous at each point of G, fix $z_0 \in G$ and suppose $\mathcal{F}$ fails to be equicontinuous at z_0. Then there is some $\varepsilon > 0$ for which no $\delta > 0$ works. Fix such an $\varepsilon > 0$ and $r > 0$ with $D(z_0, r) \subset G$. Consider the fact that $\delta = r/n$ does not work. This means there exists z_n with $|z_n - z_0| < r/n$ (hence $z_n \in G$) and $f_n \in \mathcal{F}$ so that $|f_n(z_n) - f_n(z_0)| \geq \varepsilon$.

As $\mathcal{F}$ is relatively compact in $C(G)$, there is a subsequence $(f_{n_j})_{j=1}^\infty$ which converges in $(C(G), \rho)$ (equivalently on compact subsets of G) to some $f \in C(G)$. As f is continuous at z_0 we know there exists $\delta_0 > 0$ so that

$$|z - z_0| < \delta_0 \Rightarrow |f(z) - f(z_0)| < \frac{\varepsilon}{3}$$

We can assume that $\delta_0 < r$ so that the closed disc $\bar{D}(z_0, \delta_0) \subset G$ is a compact subset of G. Thus $f_{n_j} \to f$ uniformly on $K = \bar{D}(z_0, \delta_0)$ and if j is large enough

$$\sup_{z \in \bar{D}(z_0, \delta_0)} |f_{n_j}(z) - f(z)| < \varepsilon/3$$

Thus if j is large enough for this to hold and for $1/n_j < \delta_0$ we have

$$\varepsilon < |f_{n_j}(z_{n_j}) - f_{n_j}(z_0)| \leq |f_{n_j}(z_{n_j}) - f(z_{n_j})| + |f(z_{n_j}) - f(z_0)| + |f(z_0) - f_{n_j}(z_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

This contradiction shows that $\mathcal{F}$ must be equicontinuous at z_0.

Definition A.2 *Let $G \subset \mathbb{C}$ be open and $\mathcal{F} \subset C(G)$ a family of continuous function on G. For $E \subset G$ a subset, we say that $\mathcal{F}$ on E if it satisfies:*

Given $\varepsilon > 0$ there exists $\delta > 0$ so that

$$f \in \mathcal{F}, z_0 \in E, z \in G, |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$$

(One might describe this better as 'equicontinuous uniformly at all points of E'.)

Lemma A.3 *Let $G \subset \mathbb{C}$ be open and $\mathcal{F} \subset C(G)$ a family of continuous function on G. If $\mathcal{F}$ is equicontinuous at each point of G and if $K \subset G$ is compact, then $\mathcal{F}$ is equicontinuous on K.*

(So equicontinuity at each point implies equicontinuity on compact subsets.)

Proof. Fix K. For each $\zeta \in K$, by equicontinuity at ζ we know there exists $\delta_\zeta > 0$ so that

$$f \in \mathcal{F}, |z - \zeta| < \delta_\zeta \Rightarrow |f(z) - f(\zeta)| < \frac{\varepsilon}{2}$$

Now $\left\{ D\left(\zeta, \frac{\delta_\zeta}{2}\right) : \zeta \in K \right\}$ is an open cover of K and so it has a finite subcover

$$K \subset D\left(z_1, \frac{\delta_{z_1}}{2}\right) \cup D\left(z_2, \frac{\delta_{z_2}}{2}\right) \cup \cdots \cup D\left(z_n, \frac{\delta_{z_n}}{2}\right)$$

Now put $\delta = \min(\delta_{z_1}/2, \delta_{z_2}/2, \ldots, \delta_{z_n}/2)$.

Take now $z_0 \in K$ and z with $|z - z_0| < \delta$. Then $z_0 \in D\left(z_i, \frac{\delta_{z_i}}{2}\right)$ for some i ($1 \leq i \leq n$) and then

$$|z - z_i| \leq |z - z_0| + |z_0 - z_i| \leq \delta + \frac{\delta_{z_i}}{2} \leq \delta_{z_i} \Rightarrow z \in D(z_i, \delta_{z_i}) \subset G.$$

So, for arbitrary $f \in \mathcal{F}$ we have $|f(z) - f(z_i)| < \varepsilon/2$ and $|f(z_0) - f(z_i)| < \varepsilon/2$. Hence

$$f \in \mathcal{F} \Rightarrow |f(z) - f(z_0)| \leq |f(z) - f(z_i)| + |f(z_0) - f(z_i)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

As the same δ works for all $z_0 \in K$, we have the result.

Proof. (of Theorem 5.20) One direction is already done in Proposition A.1 and so what remains is to show that if $\mathcal{F} \subset C(G)$ is both pointwise bounded and equicontinuous at each point of G, then $\mathcal{F}$ is relatively compact.

Take $\mathcal{F}$ satisfying the two conditions and $(f_n)_{n=1}^\infty$ a sequence in $\mathcal{F}$. To find the appropriate subsequence of functions we use what is known as a diagonal argument.

First we pick a countable dense subset of points of G. For example $S = \{z \in G : \Re z \in \mathbb{Q} \text{ and } \Im z \in \mathbb{Q}\}$ is countable and dense in G. To say that S is countable means we can arrange all its points in a sequence $S = \{s_1, s_2, \ldots\}$ (and dense in G means that its closure relative to G is all of G, or that each point of G is a limit of some sequence of points of S).

Now for the diagonal argument. It involves choosing subsequences of $(f_n)_{n=1}^\infty$, then further subsequences of the subsequence, and so on forever. To avoid more and more subscripts, we use $(f_{1,j})_{j=1}^\infty$ rather than $(f_{n_j})_{j=1}^\infty$ for the first subsequence, then $(f_{2,j})_{j=1}^\infty$ for the second subsequence and so on. To get off on the right track, we let $f_{0,n} = f_n$.

Since $\mathcal{F}$ is pointwise bounded,

$$\sup_{f \in \mathcal{F}} |f(s_1)| < \infty \Rightarrow \{f(s_1) : f \in \mathcal{F}\} \subset \mathbb{C} \text{ is relatively compact}$$

Thus $\{f_{0,n}(s_1) : n = 1, 2, \ldots\}$ is a relatively compact (or bounded) subset of $\mathbb{C}$. Hence the sequence $(f_{0,n}(s_1))_{n=1}^\infty$ has a subsequence $(f_{1,j}(s_1))_{j=1}^\infty$ that converges to some limit in $\mathbb{C}$. So

$$\exists \lim_{j \to \infty} f_{1,j}(s_1) \in \mathbb{C}$$

Next $\{f_{1,n}(s_2) : n = 1, 2, \ldots\}$ is relatively compact in $\mathbb{C}$ and so the sequence $(f_{1,n}(s_2))_{n=1}^\infty$ has a subsequence $(f_{2,j}(s_2))_{j=1}^\infty$ that converges to some limit in $\mathbb{C}$. So

$$\exists \lim_{j \to \infty} f_{2,j}(s_2) \in \mathbb{C}$$

Continuing in this way, once we have $(f_{n,j})_{j=1}^\infty$, we choose a subsequence $(f_{n+1,k})_{k=1}^\infty$ so that

$$\exists \lim_{k \to \infty} f_{n+1,k}(s_{n+1}) \in \mathbb{C}$$

The diagonal argument is now to choose $g_n = f_{n,n}$. Then $(g_j)_{j=n}^\infty$ is a subsequence of $(f_{n,k})_{k=1}^\infty$ and so

$$\lim_{j \to \infty} g_j(s_n) = \lim_{k \to \infty} f_{n,k}(s_n) \text{ exists in } \mathbb{C}$$

We claim that $f(z) = \lim_{j \to \infty} g_j(z)$ exists for all $z \in G$, that $f \in C(G)$ and that $g_j \to f$ in $(C(G), \rho)$ as $j \to \infty$.

Fix $z \in G$ and we claim that $(g_j(z))_{j=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$, hence convergent. Let $\varepsilon > 0$. Then, using the assumption that $\mathcal{F}$ is equicontinuous at z, there exists $\delta > 0$ so that $f \in \mathcal{F}, |\zeta - z| < \delta \Rightarrow |f(\zeta) - f(z)| < \varepsilon/3$. Note that this applies with $f = g_j$ since $g_j = f_{j,j}$ is one of the $f_n \in \mathcal{F}$. By density, there exists $s_n \in S$ with $|s_n - z| < \delta$. Then, since

$$\exists \lim_{j \to \infty} g_j(s_n) \in \mathbb{C}$$

the sequence $(g_j(s_n))_{j=1}^\infty$ is Cauchy in $\mathbb{C}$ and so there is j_0 so that

$$j, k \geq j_0 \Rightarrow |g_j(s_n) - g_k(s_n)| < \varepsilon/3$$

Thus

$$\begin{array}{rcl} j, k \geq j_0 \Rightarrow & & \\ |g_j(z) - g_k(z)| & \leq & |g_j(z) - g_j(s_n)| + |g_j(s_n) - g_k(s_n)| + |g_k(s_n) - g_k(z)| \\ & < & \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{array}$$

We now have $(g_j(z))_{j=1}^\infty$ a Cauchy sequence in $\mathbb{C}$ for each $z \in G$ and so we can define $f \colon G \to \mathbb{C}$ by

$$f(z) = \lim_{j\to\infty} g_j(z)$$

To show $g_j \to f$ uniformly on compact subsets of G, fix $K \subset G$ compact and $\varepsilon > 0$. Then there is a compact subset $K_1 \subset G$ so that $K \subset (K_1)^\circ$. (For example, using an exhaustive sequence of compact subsets of G we can show this.)

Use equicontinuity of $\mathcal{F}$ on K_1 to find $\delta > 0$ so that

$$z \in G, z_0 \in K_1, f \in \mathcal{F}, |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \frac{\varepsilon}{4}$$

Now, for each $z \in K$, since S is dense in G, there is some $s \in S \cap (B_d(z, \delta) \cap (K_1)^\circ)$. Turning this around $z \in B_d(s, \delta)$ for some $s \in S \cap (K_1)^\circ$. We can say then that

$$\{B_d(s, \delta) : s \in S \cap (K_1)^\circ\}$$

is an open cover of K. Thus there is a finite subcover

$$K \subset \bigcup_{n=1}^{n_0} B_d(s'_n, \delta)$$

for some $s'_1, s'_2, \ldots, s'_{n_0} \in S \cap (K_1)^\circ$.

Since $f(s'_n) = \lim_{j\to\infty} g_j(s'_n)$ for each s'_n, there is a j_0 so that

$$j > j_0 \Rightarrow \max_{1\leq n\leq n_0} |g_j(s'_n) - f(s'_n)| < \frac{\varepsilon}{4}$$

If we now take $z \in K$, there is some s'_n with $z \in B_d(s'_n, \delta)$ $(1 \le n \le n_0)$. We have, for $j > j_0$,

$$
\begin{aligned}
|g_j(z) - g_j(s'_n)| &< \frac{\varepsilon}{4} \\
|g_j(s'_n) - f(s'_n)| &< \frac{\varepsilon}{4} \\
|g_j(z) - f(s'_n)| &\le |g_j(z) - g_j(s'_n)| + |g_j(s'_n) - f(s'_n)| \\
&< \frac{2\varepsilon}{4} \\
|f(z) - f(s'_n)| &= \lim_{j\to\infty} |g_j(z) - f(s'_n)| \\
&\le \frac{2\varepsilon}{4} \\
|g_j(z) - f(z)| &\le |g_j(z) - f(s'_n)| + |f(z) - f(s'_n)| \\
&< \frac{2\varepsilon}{4} + \frac{2\varepsilon}{4} = \epsilon
\end{aligned}
$$

This is true for each $z \in K$ and so

$$j > j_0 \Rightarrow \sup_{z\in K} |g_j(z) - f(z)| = d_K(g_j, f) < \varepsilon$$

This means $g_j \to f$ uniformly on K, for each K.

Hence $f \in C(G)$ and $\lim_{j\to\infty} g_j = f$ in $(C(G), \rho)$.

B Baire Category Theorem and Proof of Osgoods Theorem

The Baire category theorem is usually stated for complete metric spaces. In our case, we can get by with using it only for compact metric spaces (which are automatically complete).

Definition B.1 *A* metric space (X, d) *is a set X together with a function (which is commonly called a distance function) $d\colon X \times X \to \mathbb{R}$ satisfying*

(i) $d(x_1, x_2) \ge 0 \quad (\forall x_1, x_2 \in X)$

(ii) $d(x_1, x_2) = d(x_2, x_1) \quad (\forall x_1, x_2 \in X)$

(iii) *(triangle inequality)* $d(x_1, x_3) \le d(x_1, x_2) + d(x_2, x_3) \quad (\forall x_1, x_2, x_3 \in X)$

(iv) $x_1, x_2 \in X, d(x_1, x_2) = 0 \Rightarrow x_1 = x_2$

Definition B.2 *A sequence $(x_n)_{n=1}^\infty$ is a metric space (X, d) (so $x_n \in X \forall n$) is called* convergent to a limit $x \in X$ *(and we write $\lim_{n\to\infty} x_n = x$) if*

$$\lim_{n\to\infty} d(x_n, x) = 0$$

Definition B.3 *A sequence $(x_n)_{n=1}^\infty$ is a metric space (X, d) is called a* Cauchy sequence *if* *gien any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that*

$$n, m > N \Rightarrow d(x_n, x_m) < \varepsilon$$

(This means all the terms of the sequence are close to each other, except for some at the start.)

Lemma B.4 *A convergent sequence $(x_n)_{n=1}^\infty$ in a metric space (X, d) is always a* Cauchy sequence.

Proof. Exercise

Definition B.5 *A metric space (X, d) is called* complete *if every Cauchy sequence in X is convergent (to some limit in X).*

Example B.6 $\mathbb{R}$ with the usual absolute value distance is complete. $\mathbb{Q}$ is not.

Definition B.7 *If (X, d) is a metric space $x_0 \in X$ and $r > 0$ then the* (open) ball of radius r about x_0 *is*

$$B_d(x_0, r) = \{x \in X : d(x, x_0) < r\}$$

If $S \subset X$ is a subset and $s \in S$, then s is called an interior point of S *if there is some ball $D_d(s, r) \subset S$ of positive radius $r > 0$ about s contained in S.*

The interior S° *of a subset $S \subset X$ is the set of all its interior points.*

A subset $U \subset S$ is called open *if $U^\circ = U$ (all its points are interior points). A subset $E \subset S$ is called* closed *if its complement $S \setminus E$ is open.*

Proposition B.8 *Let (X, d) be a metric space.*

(i) arbitrary unions $\bigcup_{i \in I} U_i$ of open subsets $U_i \subset X$ ($i \in I$ = any index set) are open.

(ii) the interior S° of any subset $S \subset X$ is open

(iii) the interior S° of any subset $S \subset X$ coincides with

$$\bigcup \{U : U \subset S, U \text{ open in } X\}$$

(iv) the interior S° of any subset $S \subset X$ is the largest open subset of X that is contained in S

(v) arbitrary intersections $\bigcap_{i \in I} E_i$ of closed subsets $E_i \subset X$ ($i \in I$ = any index set) are closed.

(vi) the empty subset $\emptyset$ and X are both open and closed.

(vii) *For each subset $S \subset X$, there is a smallest closed subset $\bar{S} \subset X$ containing S. It is called the* closure of S *and can be given as*

$$\bar{S} = \bigcap\{E : E \subset S, E \text{ closed in } X\}$$

or as the complement of the interior of the complement:

$$\bar{S} = X \setminus (X \setminus S)^\circ$$

Proof. Exercise.

Definition B.9 *Let (X, d) be a metric space and $K \subset X$ a subset. An* open cover *of K is any family $\mathcal{U} = \{U_i : i \in I\}$ of open subsets $U_i \subset X$ such that $K \subset \bigcup_{i \in I} U_i$ (that is, K is contained in their union).*

A subcover of a cover $\mathcal{U}$ of K is a smaller family $\mathcal{V} \subset \mathcal{V}$ so that $K \subset \bigcup \mathcal{V} = \bigcup\{U : U \in \mathcal{V}\}$. A finite subcover *is a subcover $\mathcal{V}$ that has only finitely many sets in it.*

A subset $K \subset X$ is called compact *if every open cover of K has a finite subcover.*

Proposition B.10 *For subsets of a metric space (X, d) closure, closedness and compactness can be characterised via limits of sequences:*

(i) *if $S \subset X$, then $x \in \bar{S} \iff$ there is a sequence $(s_n)_{n=1}^\infty$ of points $s_n \in S$ with $\lim_{n\to\infty} s_n = x$.*

(ii) *if $S \subset X$, then S is closed if and only if every sequence $(s_n)_{n=1}^\infty$ of points $s_n \in S$ that converges in (X, d) has its limit in S.*

(iii) *if $K \subset S$, then K is compact if and only if every sequence $(x_n)^\infty$ of points $x_n \in K$ has a subsequence $(x_{n_j})_{j=1}^\infty$ which converges to a limit in K.*

Proof. Omitted.

Proposition B.11 *Let (X, d) be a metric space. Then (X, d) is complete if and only if each Cauchy sequence in X has a convergent subsequence.*

Proof. Exercise. It is not hard to show that for a Cauchy sequence with a convergent subsequence, the whole sequence must converge (to the same limit as the subsequence).

Corollary B.12 *Compact metric spaces are complete.*

Definition B.13 *A subset $S \subset X$ of a metric space (X, d) is called* nowhere dense *if the interior of its closure is empty, $(\bar{S})^\circ = \emptyset$.*

A subset $E \subset X$ is called of first category *if it is a countable union of nowhere dense subsets, or equivalently, the union $E = \bigcup_{n=1}^\infty S_n$ of a sequence of nowhere dense sets ($(\bar{S}_n)^\circ = \emptyset \forall n$).*

A subset $Y \subset X$ is called of second category *if it fails to be of first category.*

Example B.14 (a) If a singleton subset $S = \{s\} \subset X$ fails to be nowhere dense, then the interior of its closure is not empty. The closure $\bar{S} = S = \{s\}$ and if that has any interior it means it contains a ball of some positive radius $r > 0$. So

$$B_d(s, r) = \{x \in X : d(x, s) < r\} = \{s\}$$

and this means that s is an isolated point of X (no points closer to it than r).

An example where this is possible would be $X = \mathbb{Z}$ with the usual distance (so $B(n, 1) = \{n\}$) and S any singleton subset. Another example is $X = D(2, 1) \cup \{0\} \subset \mathbb{C}$ (with the distance on X being the same as the usual distance between points in $\mathbb{C}$) and $S = \{0\}$.

(b) In many cases, there are no isolated points in X, and then a one point set is nowhere dense. So is a countable subset is then of first category ($S = \{s_1, s_2, \ldots\}$ where the elements can be listed as a finite or infinite sequence).

For example $S = \mathbb{Z}$ is of first category as a subset of $\mathbb{R}$, though it of second category as a subset of itself. $S = \mathbb{Q}$ is of first category both in $\mathbb{R}$ and in itself (because it is countable and points are not isolated).

The idea is that first countable means 'small' in some sense, while second category is 'not small' in the same sense. While it is often not hard to see that a set is of first category, it is harder to see that it fails to be of first category. One has to consider all possible ways of writing the set as a union of a sequence of subsets.

Theorem B.15 (Baire Category) *Let (X, d) be a complete metric space. Then the whole space $S = X$ is of second category in itself.*

Proof. If not, then X is of first category and that means $X = \bigcup_{n=1}^{\infty} S_n$ where each S_n is a nowhere dense subset $S_n \subset X$ (with $(\bar{S}_n)^\circ = \emptyset$).

The argument may be simplified if we assume each S_n is closed (which we can do if we replace S_n by its closure) but we will just continue with $\bar{S}_n$.

Since $\bar{S}_n$ has empty interior, its complement is a dense open set. That is

$$\overline{X \setminus \bar{S}_n} = X \setminus (\bar{S}_n)^\circ = X$$

Thus if we take any ball $B_d(x, r)$ in X, there is a point $y \in (X \setminus \bar{S}_n) \cap B_d(x, r)$ and then because $X \setminus \bar{S}_n$ is open there is a (smaller) $\delta > 0$ with $B_d(y, \delta) \subset (X \setminus \bar{S}_n) \cap B_d(x, r)$.

Start with $x_0 \in X$ any point and $r_0 = 1$. Then, by the above reasoning there is a ball $B_d(x_1, r_1) \subset (X \setminus \bar{S}_1) \cap B_d(x_0, r_0)$. In fact, making r_1 smaller if necessary, we can ensure that the closed ball

$$\bar{B}_d(x_1, r_1) = \{x \in X : d(x, x_1) \leq r_1\} \subset (X \setminus \bar{S}_1) \cap B_d(x_0, r_0)$$

and $r_1 < 1$. We can then find x_2 and $r_2 \leq r_1/2 < 1/2$ so that

$$\bar{B}_d(x_2, r_2) \subset (X \setminus \bar{S}_2) \cap B_d(x_1, r_1)$$

and we can continue this process to select $x_1, x_2, \ldots$ and $r_1, r_2, \ldots$ with

$$0 < r_n \leq r_{n-1}/2 < \frac{1}{2^{n-1}}, \quad \bar{B}_d(x_n, r_n) \subset (X \setminus \bar{S}_n) \cap B_d(x_{n-1}, r_{n-1}) \quad (n = 1, 2, \ldots)$$

We claim the sequence $(x_n)_{n=1}^\infty$ is a Cauchy sequence in X. This is because $m \geq n \Rightarrow x_m \in B_d(x_n, r_n) \Rightarrow d(x_m, x_n) < r_n < 1/2^n$. So, if n, m are both large

$$d(x_m, x_n) < \min\left(\frac{1}{2^n}, \frac{1}{2^m}\right)$$

is small.

By completeness, $x_\infty = \lim_{n\to\infty} x_n$ exists in X. Since the closed ball $\bar{B}_d(x_n, r_n)$ is a closed set in X and contains all x_m for $m \geq n$, it follows that $x \in \bar{B}_d(x_n, r_n)$ for each n. But $\bar{B}_d(x_n, r_n) \subset X \setminus \bar{S}_n$ and so $x \notin \bar{S}_n$. This is true for all n and so we have the contradiction

$$x \notin \bigcup_{n=1}^\infty \bar{S}_n = X$$

Thus X cannot be a union of a sequence of nowhere dense subsets.

Corollary B.16 *Let (X, d) be a compact metric space. Then the whole space $S = X$ is of second category in itself.*

Proof. Compact metric spaces are complete. So this follows from the theorem.
Proof. (of Osgoods Theorem 5.22)

As outlined previously, take

$$S_m = \{z \in G : \sup_n |f_n(z)| \leq m\} = \bigcap_{n=1}^\infty \{z \in G : |f_n(z)| \leq m\}$$

$G_m = (S_m)^\circ$ the interior of S_m and $G_0 = \bigcup_{m=1}^\infty G_m$.

Clearly G_0 is open.

Note that $z \in G \Rightarrow \lim_{n\to\infty} f_n(z) = f(z) \in \mathbb{C}$ and so the sequence $(f_n(z))_{n=1}^\infty$ must be bounded. If m is big enough $z \in S_m$ and so we have $\bigcup_{m=1}^\infty S_m = G$.

To show that G_0 is dense in G, fix $z \in G$ and a disc $D(z, r)$ about z of small enough radius that its closure $\bar{D}(z, r) \subset G$. Then $\bar{D}(z, r)$ is a compact metric space and

$$\bar{D}(z, r) \subset G = \bigcup_{m=1}^\infty S_m \Rightarrow \bar{D}(z, r) = \bigcup_{m=1}^\infty S_m \cap \bar{D}(z, r)$$

Applying the Baire category theorem to the compact metric space $\bar{D}(z, r)$ we find there is m so that $S_m \cap \bar{D}(z, r)$ is not nowhere dense in $\bar{D}(z, r)$. As $S_m \cap \bar{D}(z, r)$ is closed, that means it has nonempty interior *as a subset of* $\bar{D}(z, r)$. There is therefore a ball center w and radius $\delta > 0$ in the metric space $\bar{D}(z, r)$ that is contained in $S_m \cap \bar{D}(z, r)$. This ball is in fact the intersection

$$D(w, \delta) \cap \bar{D}(z, r)$$

of an open and a closed disc. Thus $D(w, \delta) \cap D(z, r)$ is not empty, open and is contained in $S_m \cap \bar{D}(z, r)$. So $D(w, \delta) \cap D(z, r) \subset (S_m)^\circ = G_m \subset G_0$ and we have

$$D(z, r) \cap G_0 \neq \emptyset$$

This shows that G_0 is dense in G.

The rest of the proof of Osgoods theorem was given earlier.