Chapter 4: Open mapping theorem, removable singularities

Course 414, 2003–04

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Theorem 4.1 (Laurent expansion) *Let $f\colon G \to \mathbb{C}$ be analytic on an open $G \subset \mathbb{C}$ be open that contains a nonempty annulus $\{z \in \mathbb{C} : R_1 < |z-a| < R_2\}$ (some $0 \leq R_1 < R_2 \leq \infty$, some center $a \in \mathbb{C}$). Then $f(z)$ can be represented by a Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \quad (R_1 < |z-a| < R_2)$$

where, for any choice of r with $R_1 < r < R_2$ and $\gamma_r\colon [0,1] \to \mathbb{C}$ given by $\gamma_r(t) = a + r\exp(2\pi i t)$

$$a_n = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{(z-a)^{n+1}}\, dz$$

Proof. Recall from 1.9 that a Laurent series has an annulus of convergence. In this case the annulus of convergence must include $R_1 < |z-a| < R_2$.

Turning to the proof, notice that from Cauchy's theorem we can conclude that the formula given for a_n is independent of r in the range $R_1 < r < R_2$. Fix any z with $R_1 < |z-a| < R_2$ and choose r_1, r_2 with $R_1 < r_1 < |z-a| < r_2 < R_2$. From the winding number version of Cauchy's integral formula (1.30 with $\Gamma = \gamma_{r_2} - \gamma_{r_1}$) we can deduce

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_{r_2}} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{\gamma_{r_1}} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

The remainder of the proof is quite like the theorem that analytic function in a disk are represented by power series (Theorem 1.23).

For $|\zeta - a| = r_2$, we do exactly as in the power series case

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a) - (z-a)} = \frac{1}{\zeta - a} \frac{1}{1 - \frac{z-a}{\zeta - a}} = \frac{1}{\zeta - a} \frac{1}{1-w}$$

where $w = \frac{z-a}{\zeta-a}$ has $|w| < \frac{|z-a|}{r_2} = \rho_2 < 1$. Hence

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a} \sum_{n=0}^{\infty} w^n = \frac{1}{\zeta - a} \sum_{n=0}^{\infty} \left(\frac{z-a}{\zeta - a}\right)^n = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\zeta - a)^{n+1}}.$$

For $|\zeta - a| = r_1$, we do something similar

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a) - (z - a)} = \frac{-1}{z - a} \frac{1}{1 - \frac{\zeta - a}{z - a}} = \frac{-1}{z - a} \frac{1}{1 - w}$$

and this time $w = \frac{\zeta - a}{z - a}$ is also less than one in modulus (for all ζ on $|\zeta - a| = r_1$). In fact $|w| = r_1/|z - a| = \rho_1 < 1$. Thus we get another series

$$\frac{1}{\zeta - z} = \frac{-1}{z - a} \sum_{n=0}^{\infty} w^n = \frac{-1}{z - a} \sum_{n=0}^{\infty} \left(\frac{\zeta - a}{z - a} \right)^n = -\sum_{n=0}^{\infty} \frac{(\zeta - a)^n}{(z - a)^{n+1}}. = -\sum_{m=1}^{\infty} \frac{(\zeta - a)^{m-1}}{(z - a)^m}.$$

Plugging these two series into the the two integrals in the integral formula for $f(z)$ and exchanging the order of integral and sum (using uniform convergence to justify the exchange) we get

$$f(z) = \sum_{n=0}^{\infty} \left(\int_{\gamma_{r_2}} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, d\zeta \right) (z - a)^n + \sum_{m=1}^{\infty} \left(\int_{\gamma_{r_1}} f(\zeta)(\zeta - a)^{m-1} \, d\zeta \right) \frac{1}{(z - a)^m}$$

and this comes down to the desired result.

Definition 4.2 *If an analytic function f is analytic on an open set $G \subset \mathbb{C}$ that includes a punctured disk $D(a, r) \setminus \{a\}$ of positive radius $r > 0$ about some $a \in \mathbb{C}$, then a is called an* (isolated) singularity *if a is not itself in G.*

The residue *of f at an isolated singularity a of f is defined as the coefficient a_{-1} in the Laurent series for f in a punctured disk about a:*

$$\textit{res}(f, a) = a_{-1} = \frac{1}{2\pi i} \int_{|z-a|=\delta} f(z) \, dz$$

if $0 < \delta <$ the radius of a punctured disk about a where f is analytic.

Theorem 4.3 (Residue theorem) *Let $G \subset \mathbb{C}$ be open and suppose f is analytic in $G \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ for some finite number of (distinct) points $\alpha_1, \alpha_2, \ldots, \alpha_n \subset G$. Suppose Γ is a (piecewise C^1) chain in $G \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ with the property that $\mathrm{Ind}_\Gamma(w) = 0$ for all $w \in \mathbb{C} \setminus G$.*

Then

$$\int_\Gamma f(z) \, dz = 2\pi i \sum_{j=1}^{n} \textit{res}(f, \alpha_j) \textit{Ind}_\Gamma(\alpha_j).$$

Proof. We apply the winding number version of Cauchy's theorem 1.30 to a new chain Γ_1 constructed as follows. Choose $\delta > 0$ so small that (i) $\bar{D}(\alpha_j, \delta) \subset G$ and (ii) $\delta < |\alpha_j - \alpha_k|$ for $j \neq k$ and $1 \leq j, k \leq n$. Let γ_j be the circle $|z - a| = \delta$ traversed $-\mathrm{Ind}_\Gamma(\alpha_j)$ times anticlockwise and let

$$\Gamma_1 = \Gamma + \gamma_1 + \gamma_2 + \cdots + \gamma_n.$$

Now Γ_1 is a new chain in $H = G \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. H is open, f is analytic in H and $\mathrm{Ind}_{\Gamma_1}(w) = 0$ for all $w \in \mathbb{C} \setminus H$. To check the last assertion notice that $\mathrm{Ind}_{\gamma_j}(w) = 0$ for all $w \in \mathbb{C} \setminus G$ and so $\mathrm{Ind}_{\Gamma_1}(w) = \mathrm{Ind}_{\Gamma}(w) + \sum_{j=1}^{n} \mathrm{Ind}_{\gamma_j}(w) = 0$ for these w. The remaining $w \in \mathbb{C} \setminus H$ are $w = \alpha_j$ ($1 \leq j \leq n$). Notice that

$$\mathrm{Ind}_{\gamma_j}(\alpha_k) = \begin{cases} -\mathrm{Ind}_{\Gamma}(\alpha_j) & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

since $\alpha_k \notin \bar{D}(\alpha_j, \delta)$ for $k \neq j$, and so

$$\mathrm{Ind}_{\Gamma_1}(\alpha_j) = \mathrm{Ind}_{\Gamma}(\alpha_j) + \sum_{k=1}^{n} \mathrm{Ind}_{\gamma_j}(\alpha_k) = \mathrm{Ind}_{\Gamma}(\alpha_j) - \mathrm{Ind}_{\Gamma}(\alpha_j) = 0.$$

By 1.30,

$$\int_{\Gamma_1} f(z)\, dz = 0$$

and this means

$$0 = \int_{\Gamma} f(z)\, dz + \sum_{j=1}^{n} \int_{\gamma_j} f(z)\, dz = \int_{\Gamma} f(z)\, dz + \sum_{j=1}^{n} 2\pi i(-\mathrm{Ind}_{\Gamma}(\alpha_j))\mathrm{res}(f, \alpha_j)$$

and the result follows.

Corollary 4.4 (Residue theorem, homotopy version) *Let $G \subset \mathbb{C}$ be open and $f(z)$ a function analytic on G except perhaps for a finite number of (distinct) points $\alpha_1, \alpha_2, \ldots, \alpha_n \in G$. (More formally, f is analytic on $G \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.) Let γ be a (piecewise C^1) closed curve in $G \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ which is null-homotopic in G. Then*

$$\int_{\gamma} f(z)\, dz = 2\pi i \sum_{j=1}^{n} \mathit{res}(f, \alpha_j) \mathit{Ind}_{\gamma}(\alpha_j).$$

Proof. We can apply Theorem 4.3 since $\mathrm{Ind}_{\gamma}(w) = 0$ for $w \in \mathbb{C} \setminus G$ by 1.43.

Corollary 4.5 (Residue theorem, simple closed curve version) *Let $G \subset \mathbb{C}$ be open and $f(z)$ a function analytic on G except perhaps for a finite number of (distinct) points $\alpha_1, \alpha_2, \ldots, \alpha_n \in G$. Suppose that γ is a (piecewise C^1) simple closed curve in $G \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, oriented anticlockwise and with its interior contained in G. Then*

$$\int_{\gamma} f(z)\, dz = 2\pi i \sum_{\{j : 1 \leq j \leq n, \alpha_j \text{ inside } \gamma\}} \mathit{res}(f, \alpha_j)$$

Proof. By definition, the outside of γ is the unbounded component of $\mathbb{C} \setminus \gamma$. Also $\mathrm{Ind}_{\gamma}(w) = 0$ for w outside γ. Since γ and its inside is contained in G, $w \in \mathbb{C} \setminus G \Rightarrow w$ outside $\gamma \Rightarrow \mathrm{Ind}_{\gamma}(w) = 0$. So we can apply the theorem (4.3) to γ.

By definition of anticlockwise, $\mathrm{Ind}_{\gamma}(\alpha_j) = 1$ if α_j is inside γ and $\mathrm{Ind}_{\gamma}(\alpha_j) = 0$ if α_j outside. Thus the formula of Theorem 4.3 for $\int_{\gamma} f(z)\, dz$ reduces to the one above in this situation.

Remark 4.6 The residue theorem can be used to work out many integrals of analytic functions along closed curves in $\mathbb{C}$. It is only necessary to be able to work out residues (and winding numbers, but in many examples winding numbers are easy to find).

To find, for example, the residue of

$$f(z) = \frac{e^z}{(z-1)^2(z-2)}$$

at $z = 1$, we can write it as

$$f(z) = \frac{1}{(z-1)^2}\frac{e^z}{z-2} = \frac{1}{(z-1)^2}g(z)$$

where $g(z)$ is analytic near $z = 1$. It follows that $g(z)$ has a power series representation in some disk about 1 (in fact in $D(1, 1)$, the largest disk that excludes $z = 2$)

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(1)}{n!}(z-1)^n$$

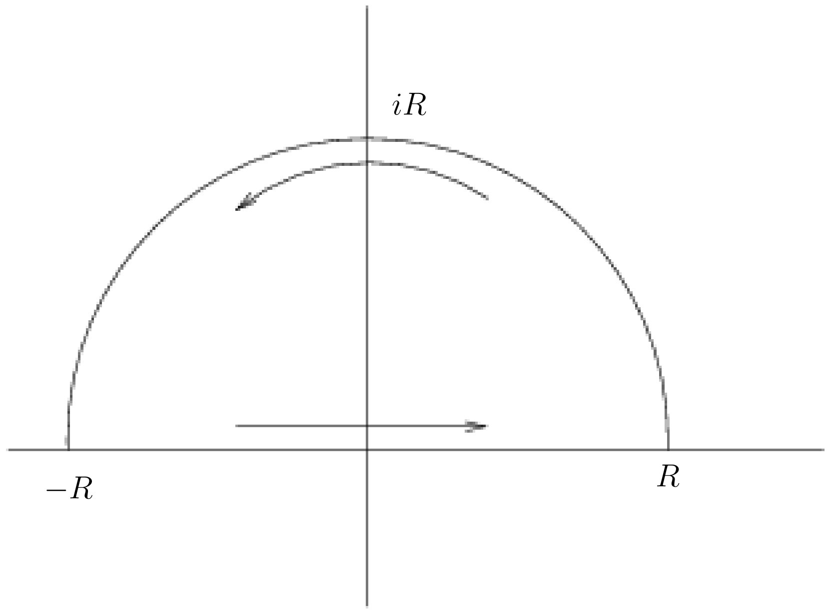
Hence

$$f(z) = \frac{g(1)}{(z-1)^2} + \frac{g'(1)}{1!(z-a)} + \frac{g^{(2)}(1)}{2!} + \cdots$$

and the coefficient of $(z-1)^{-1}$ is the residue. So it is $g'(1) = \frac{e(1-2)-e}{(1-2)^2} = -2e$.

The residue theorem is also an effective technique for working out certain real integrals. We will not go into this in any detail, but here are some examples that give the flavour of the methods used.

Consider $\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx$. One can verify that the integral converges by comparison with $\int_1^{\infty} \frac{1}{x^2}\,dx$ or $\int_{-\infty}^{\infty} \frac{2}{x^2+1}\,dx$. Consider the complex analytic function $f(z) = \frac{z^2}{z^4+1}$ and the closed curve made up of the real axis from $-R$ to R followed by the semicircle of radius R in the upper half plane (say) from R back to $-R$. We take R large (at least $R > 1$). By the residue theorem, we can work out this complex integral and get $2\pi i$ times the sum of the residues at $z = e^{i\pi/4}$ and $z = e^{3\pi i/4}$.



These residues are in fact $\frac{i}{4}\exp(-3\pi i/4)$ and $\frac{-i}{4}\exp(-\pi i/4)$. $2\pi i$ times their sum is $\pi/\sqrt{2}$. So if σ_R denotes the semicircle of radius R, then

$$\int_{-R}^{R} \frac{x^2}{x^4+1}\,dx + \int_{\sigma_R} \frac{z^2}{z^4+1}\,dz = \frac{\pi}{\sqrt{2}}$$

For large R, the integral over the semicircular part of the contour is at most the length of the contour (πR) times the maximum value of the integrand, or

$$\left|\int_{\sigma_R} \frac{z^2}{z^4+1}\,dz\right| \leq \pi R \frac{R^2}{R^4-1} \to 0 \text{ as } R \to \infty$$

Using this we find that

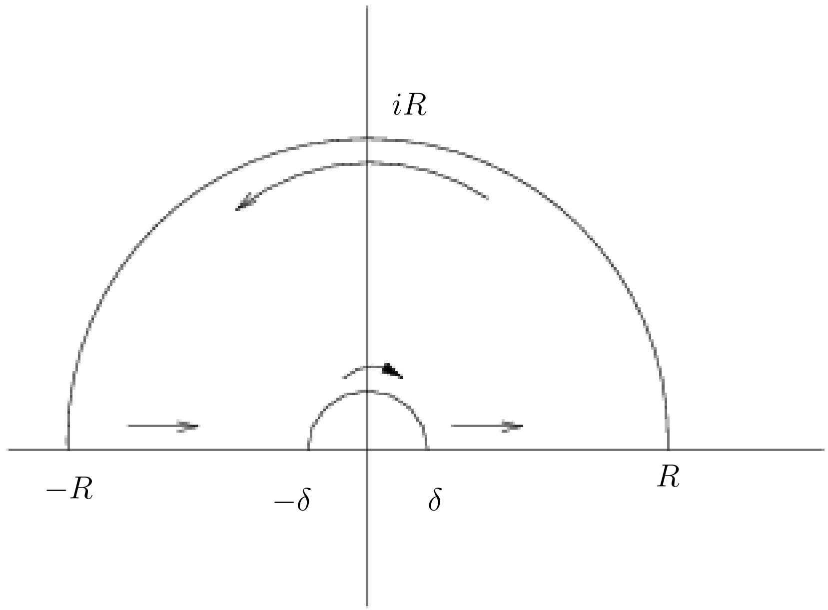
$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{x^4+1}\,dx = \frac{\pi}{\sqrt{2}}$$

and so $\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx = \pi/\sqrt{2}$.

Another fairly standard example which can be worked out is $\int_0^\infty \frac{\sin x}{x}\,dx$. The contour to use is the line segment $-R$ to $-\delta$ (R big, δ small, both positive), the semicircle of radius δ around the origin from $-\delta$ to δ (say the one above the real axis and denote it by σ_δ), the line segment δ to R and the same semicircle σ_R as above. If this closed curve is γ, then

$$\int_\gamma \frac{e^{iz}}{z}\,dz = 0$$

by Cauchy's theorem.



Thus

$$\int_{-R}^{-\delta} \frac{e^{ix}}{x}\,dx + ` \int_{\sigma_\delta} \frac{e^{iz}}{z}\,dz + \int_{\delta}^{R} \frac{e^{iz}}{z}\,dz + \int_{\sigma_R} \frac{e^{ix}}{x}\,dx = 0$$

One can show (but it is a bit trickier this time) that the integral around the big semicircle σ_R tends to zero as $R \to \infty$, while (taking $z = \delta \exp(i(\pi - t))$, $0 \leq t \leq \pi$ as a parametrisation of σ_δ)

$$\lim_{\delta \to 0} \int_{\sigma_\delta} \frac{e^{iz}}{z}\,dz = \lim_{\delta \to 0} \int_0^\pi \frac{\exp(i\delta \exp(i(\pi - t)))}{\delta \exp(i(\pi - t))} \delta(-i) \exp(i(\pi - t))\,dt = -i\pi$$

and

$$\int_{-R}^{-\delta} \frac{e^{ix}}{x}\,dx + \int_{\delta}^{R} \frac{e^{ix}}{x}\,dx = 2i \int_{\delta}^{R} \frac{\sin x}{x}\,dx.$$

Taking limits, one finally ends up with

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Definition 4.7 *Let $f(z)$ be analytic on some open set $G \subset \mathbb{C}$. A point $a \in G$ is called a* zero of f *if $f(a) = 0$. The point a is called a* zero of f of multiplicity m *(for $m > 0$ a positive integer) if $f(a) = 0$ and $f^{(j)}(a) = 0$ for $1 \leq j < m$ but $f^{(m)}(a) \neq 0$.*

Equivalently, if we look at a power series representation of f is a disk about a, the first term with a nonzero coefficient is the $(z - a)^m$ term.

$$f(z) = a_m(z - a)^m + a_{m+1}(z - a)^{m+1} + \cdots = \sum_{n=m}^{\infty} a_n (z - a)^n$$

with $a_m = f^{(m)}(a)/m! \neq 0$.

When we look at the case of a polynomial $p(z) = b_0 + b_1 z + \cdots + b_n z^n$ of degree n (so $b_n \neq 0$ then we know from the Fundamental theorem of algebra (3.11) that we can factor

$$p(z) = b_n(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$$

Thus $p(z)$ has the roots α_j for $1 \leq j \leq n$ and we can say $p(z)$ has at most n roots. We cannot say it has exactly n because there may be repetitions among the α_j. If we group the like terms

$$p(z) = b_n(z - \beta_1)^{m_1}(z - \beta_2)^{m_2} \cdots (z - \beta_k)^{m_k}$$

with $\beta_1, \beta_2, \ldots, \beta_k$ the distinct roots and $m_1 + m_2 + \cdots + m_k = n$. Then m_j is the multiplicity of the zero β_j. What we see is that is if we count each zero β_j as many times as its multiplicity m_j, then we can say that every polynomial $p(z)$ of degree n has exactly n roots.

Theorem 4.8 (Argument principle, simple version) *Let $G \subset \mathbb{C}$ be simply connected and $f \colon G \to \mathbb{C}$ analytic and not identically zero in G. Let γ be an anticlockwise simple closed curve in G and assume that $f(z)$ is never zero on γ. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the zeros of f that are inside γ and let α_j have multiplicity m_j ($1 \leq j \leq n$).*

Then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = m_1 + m_2 + \cdots + m_n = N$$

(= the total number of zeros of f inside γ counting multiplicities, something we will often denote by N or N_γ or $N_{\gamma,f}$).

Proof. There are some aspects of the statement that may require some elaboration. Since G is simply connected, we can say that if $w \in \mathbb{C} \setminus G$, then

$$\int_\gamma \frac{1}{z - w}\, dz = 0$$

by Cauchy's theorem (in the form 2.3 for a simply connected G — using $1/(z - w)$ analytic in G). Hence $\mathrm{Ind}_\gamma(w) = 0$ for all $w \in \mathbb{C} \setminus G$ and so the inside of γ (where the index is $+1$) must be in G as well as γ itself.

Now γ together with its inside is a compact subset of $\mathbb{C}$, contained in G. So, by the identity theorem (Corollary 3.5), there can only be a finite number of zeros of f inside or on γ. We have assumed there are non on γ itself.

Finally, each zero of f has a finite multiplicity (again by the identity theorem 3.1) since G is connected and f is not identically zero.

These are all points that are implicit in the statement of the theorem (or without which we would need further assumptions in order for the statement to make sense). The proof of the theorem is essentially to apply the residue theorem and to show that the residue of f'/f at a zero α_j is the multiplicity m_j.

There is a small catch as the statement of the residue theorem we would like to rely upon (4.5) is stated with a finite total number of singularities. In our case f'/f has singularities where

$f(z) = 0$ and there could be infinitely many such points in all of G. What we can do is shrink G to a smaller open set H that still contains γ and its interior but where f has only finitely many zeros.

One way to do that is to construct H so that its closure is compact and contained in G. Let K denote the union of γ with its inside (already noted to be a compact subset of G). For each $z \in K$ we can find a $\delta_z > 0$ so that the closed disk $\bar{D}(z, \delta_z) \subset G$. Then $K \subset \bigcup_{z \in K} D(z, \delta_z)$ is an open cover of K and so has a finite subcover (K compact) $K \subset \bigcup_{j=1}^k D\left(z_j, \delta_{z_j}\right)$. Take $H = \bigcup_{j=1}^k D\left(z_j, \delta_{z_j}\right)$ and then H is open, $\bar{H} \subset \bigcup_{j=1}^k \bar{D}\left(z_j, \delta_{z_j}\right) \subset G$ is a compact subset of G. By the identity theorem, f has only finitely many zeros in $\bar{H}$, hence only finitely many in H. Thus we can apply the residue theorem (4.5) to get

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 2\pi i \sum_{j=1}^n \text{res}\left(\frac{f'}{f}, \alpha_j\right)$$

To complete the proof we show that if $f(z)$ has a zero $z = \alpha$ of multiplicity m then res$(f, \alpha) = m$. To do this, start with a power series for f in a disk about α

$$f(z) = \sum_{k=m}^\infty \frac{f^{(k)}(\alpha)}{k!}(z-\alpha)^k = \sum_{k=m}^\infty a_k (z-\alpha)^k$$

with $a_m = f^{(k)}(\alpha)/k! \neq 0$ (and this expansion is valid in some disk $|z - \alpha| < \delta$). We also have

$$f'(z) \sum_{k=m}^\infty k a_k (z-\alpha)^{k-1}$$

(in the same disk $|z - \alpha| < \delta$). Hence

$$\frac{f'(z)}{f(z)} = \frac{(z-\alpha)^{m-1} \sum_{k=m}^\infty k a_k (z-\alpha)^{k-m}}{(z-a)^m \sum_{k=m}^\infty a_k (z-\alpha)^{k-m}} = \frac{1}{z-\alpha} g(z)$$

where $g(z)$ is analytic in some small disk about α (a small enough disk where the denominator is never 0, exists because the denominator of $g(z)$ is $a_m \neq 0$ at $z = \alpha$ and so it remains nonzero in some disk about α by continuity).

Now $g(z)$ has a power series around α

$$g(z) = \sum_{k=0}^\infty \frac{g^{(k)}(\alpha)}{k!}(z-\alpha)^k$$

with $g(\alpha) = \frac{m a_m}{a_m} = m$ and so $f'(z)/f(z)$ has a Laurent series

$$\frac{f'(z)}{f(z)} = \frac{g(z)}{z-\alpha} = \sum_{k=0}^\infty \frac{g^{(k)}(\alpha)}{k!}(z-\alpha)^{k-1}$$

where the coefficient of $(z-a)^{-1}$ is $g(0) = m$. So the residue of f'/f at $z = \alpha$ is m, as claimed.

This completes the proof.

Remark 4.9 One might ask why this theorem is called the argument principle. If we consider the composition of f with the curve γ we get a new curve $f \circ \gamma$ in $\mathbb{C} \setminus \{0\}$ (because $f(z)$ is never zero on γ). If we compute the index of this curve around the origin, we get

$$\mathrm{Ind}_{f\circ\gamma}(0) = \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{1}{w}\,dw = \frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}\,dz$$

(as can be seen by writing the integrals in terms of a parameter $w = f(z) = f(\gamma(t))$).

Thus the integral in the theorem is the number of times $f(z)$ goes anticlockwise around the origin as z goes around γ.

Theorem 4.10 (Open mapping theorem) *Let $G \subset \mathbb{C}$ be a connected open set and $f \colon G \to \mathbb{C}$ analytic but not constant. Then for each open subset $U \subset G$ the image $f(U)$ is open in $\mathbb{C}$.*

(That means that forward images of open sets are open, while inverse images of open sets are open by continuity.)

Proof. Fix $U \subset G$ open and $w_0 \in f(U)$. Thus $w_0 = f(z_0)$ for some $z_0 \in U$. (This z_0 may not be unique, but fix one.) Now the function $f(z) - w_0$ has a zero at $z = z_0$. As G is connected and f is not constant ($f(z) - w_0$ is not identically zero on G) the identity theorem (3.1) tells us that this zero has a finite multiplicity $m \geq 1$.

There must be some $\delta > 0$ with $f(z) - w_0$ never zero for $0 < |z - z_0| < \delta$ (and $D(z_0, \delta) \subset G$) by the identity theorem again. Choose $r < \delta$, $r > 0$ with $\bar{D}(z_0, r) \subset U$. Then, by the argument principle (4.8) we must have

$$\frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f'(z)}{f(z) - w_0}\,dz = m$$

(= the total number of zeros of $f(z) - w_0$ inside $|z - z_0| = r$ counting multiplicities).

Next $|f(z) - w_0|$ is a real-valued function which is continuous and always strictly positive on the compact circle $|z - z_0| = r$. Hence it has a minimum value $\varepsilon > 0$ and

$$\inf_{|z-z_0|=r} |f(z) - w_0| = \varepsilon > 0.$$

Now if $|w - w_0| < \varepsilon$, then $|f(z) - w| \geq |f(z) - w_0| - |w - w_0| \geq \varepsilon - |w - w_0| > 0$ on $|z - z_0| = r$ and so

$$\frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f'(z)}{f(z) - w}\,dz = N(w)$$

gives the total number of solutions of $f(z) - w = 0$ inside the circle $|z - z_0| = r$.

As a function of w, $N(w)$ is a continuous function of w for $|w - w_0| < \varepsilon$. It is an integer-valued continuous function on the connected disk $D(w_0, \varepsilon)$. It is therefore constant $N(w) = N(w_0) = m$.

As $m > 1$, this means that if we take any $w \in D(w_0, \varepsilon)$ then there is at least one $z \in D(z_0, r)$ with $f(z) = w$. In other words $D(w_0, \varepsilon) \subset f(D(z_0, r)) \subset f(U)$. Hence w_0 is an interior point of $f(U)$. True for all $w_0 \in f(U)$ and so $f(U)$ is open.

Corollary 4.11 (Inverse function theorem) *If $G \subset \mathbb{C}$ is open and $f\colon G \to \mathbb{C}$ is an injective analytic function, then*

(i) *$f(G)$ is open*

(ii) *$f'(z)$ is never zero in G*

(iii) *the inverse function $f^{-1}\colon f(G) \to G \subset \mathbb{C}$ is analytic and its derivative is*

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}$$

In other words: if $w = f(z)$ then the inverse $z = f^{-1}(w)$ has derivative

$$\frac{dz}{dw} = \frac{1}{\frac{dw}{dz}}$$

Proof.

(i) Note that G is the union of its connected components $G = \bigcup_{i \in I} G_i$. Now each restriction of f to a connected component G_i is injective and analytic on the open set G_i. Hence, by Theorem 4.10, $f(G_i)$ is open. Hence $f(G) = \bigcup_{i \in I} f(G_i)$ is open.

(ii) If $f'(z_0) = 0$ for some $z_0 \in G$, then we can use the arguments of the proof of Theorem 4.10 with $m > 1$. we find out that there is some $\varepsilon > 0$ so that for $|w - f(z_0)| < \varepsilon$ we have $N(w) = m > 0$ solutions for $f(z) - w = 0$ counting multiplicities and only looking at z's inside a small disk $D(z_0, r)$. We can make this claim as long as $r > 0$ is small enough and then $\varepsilon > 0$ is chosen to depend on r.

Now $f'(z)$ analytic but not identically zero in the connected component of z_0 in G (reason: if f' was identically zero there, then $f(z)$ would have to be constant there and that would mean it was not injective). So, by the identity theorem (3.1 applied to f' on the connected component) we can choose $r > 0$ small enough that $f'(z)$ is never zero for $0 < |z - z_0| < r$. This means that for this or smaller r there are no multiple zeros of $f(z) - w$ because there are no points were $f'(z) = 0$ except $z = z_0$ and $w = w_0$. Hence if we take $w \neq w_0$ there are $m > 1$ distinct solutions of $f(z) - w = 0$. This contradicts injectivity of f. So f' is never zero.

(iii) Certainly the inverse map $f^{-1}\colon f(G) \to G$ makes sense. To show it is analytic, we work on each connected component G_i of G separately and consider the restriction f_i of f to G_i and the corresponding inverse $f_i^{-1}\colon f(G_i) \to G_i$ (which is the restriction of f^{-1} to $f(G_i)$). In other words, we can deduce the result if we prove it for the case where G is connected open.

So we assume from now on that G is connected. By the Open Mapping Theorem 4.10, forward images $f(U)$ of open subsets of G are open. But this means that f^{-1} is continuous because the inverse image under the inverse function

$$(f^{-1})^{-1}(U) = f(U)$$

and is therefore open for $U \subset G$ open.

Now we can directly compute the derivative of f^{-1} at a point $w_0 \in f(G)$ as follows. Let $z_0 = f^{-1}(w_0)$ and then

$$\begin{aligned}(f^{-1})'(w_0) &\stackrel{def}{=} \lim_{k\to 0} \frac{f^{-1}(w_0+k) - f^{-1}(w_0)}{k} \\ &= \lim_{k\to 0} \frac{h}{f(z_0+h) - f(z_0)}\end{aligned}$$

where we define $h = h(k) = f^{-1}(w_0+k) - f^{-1}(w_0) = f^{-1}(w_0+k) - z_0$. By continuity of f^{-1} we can say that $\lim_{k\to 0} h(k) = 0$ and because of bijectivity of f^{-1} we can say that for $k \neq 0$ small enough $h = h(k) \neq 0$.

As $k \to 0$ we have

$$\frac{f(z_0+h) - f(z_0)}{h} \to f'(z_0) \neq 0$$

and so the reciprocal

$$\frac{h}{f(z_0+h) - f(z_0)} \to \frac{1}{f'(z_0)}$$

Hence $(f^{-1})'(w_0)$ exists and is $1/f'(f^{-1}(w_0))$.

Example 4.12 Since the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}$ is analytic, if $G \subset \mathbb{C}$ is any open set where $\exp$ is injective (equivalently where it is not possible to have $z_1, z_2 \in G$, $z_1 \neq z_2$ and $z_1 = z_2 + 2n\pi$, $n \in \mathbb{Z}$) then the restriction $\exp|_G$ of the map to G has an inverse

$$(\exp|_G)^{-1} : \exp(G) \to G \subset \mathbb{C}$$

which is analytic. This will be a branch of $\log w$ for $w \in \exp(G)$ since $\exp\left((\exp|_G)^{-1}(w)\right) = w \forall w \in \exp(G)$.

If we take G to be the strip $G = \{z \in \mathbb{C} : -\pi < \Im(z) < \pi\}$ we have

$$\exp(G) = \left\{e^{x+iy} : -\pi < y < \pi\right\} = \{w \in \mathbb{C} : w \text{ not a negative real number}\} = \mathbb{C} \setminus (-\infty, 0]$$

and so the inverse function in this case is the principal branch $\operatorname{Log} w$.

Theorem 4.13 *If an analytic function $f(z)$ has an isolated singularity $z = a$ and*

$$\sup_{0<|z-a|<\delta} |f(z)| < \infty$$

for some $\delta > 0$ (that is if f is bounded in some punctured disc about a), then there exists an analytic extension of $f(z)$ to include $z = a$.

That is, if $f\colon G \to \mathbb{C}$ is analytic on $G \subset \mathbb{C}$ open and $a \in \mathbb{C} \setminus G$ satisfies the hypotheses, then there exists $g\colon G \cup \{a\} \to \mathbb{C}$ analytic with $g(z) = f(z) \forall z \in G$ (and $g(a) = \lim_{z\to a} f(z)$).

Proof. Consider the Laurent series for f is a punctured disc about a,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \quad (0 < |z-a| < \delta)$$

where

$$a_n = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{f(z)}{(z-a)^{n+1}}\, dz$$

(any $0 < r < \delta$). Let $M = \sup_{0<|z-a|<\delta} |f(z)|$ and estimate

$$|a_n| \leq \frac{1}{2\pi}(2\pi r) \sup_{|z-a|=r} \frac{|f(z)|}{|z-a|^{n+1}} \leq r \frac{M}{r^{n+1}} = M r^{-n}$$

We have this estimate for all small $r > 0$. If $n < 0$ (and so $-n > 0$), let $r \to 0^+$ to get $|a_n| = 0$ for all $n < 0$. Thus the Laurent series for f is in fact a power series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n = \sum n = 0^\infty a_n (z-a)^n \quad (0 < |z-a| < \delta)$$

If we define $g(a) = a_0$ and $g(z) = f(z)$ for all other z where $f(z)$ is analytic, then we get g analytic everywhere where f was and also $g(z) = \sum_{n=0}^\infty a_n (z-a)^n$ for $|z-a| < \delta$ shows that g is analytic at $z = a$ also.

Corollary 4.14 *If $f(z)$ is analytic with an isolated singularity at $z = a$, then $z = a$ is a removable singularity (meaning that f can be extended to $z = a$ so as to make it analytic there)* $\iff$

$$\lim_{z \to a} (z-a) f(z) = 0$$

Proof. $\Rightarrow$: If there is an extension, then $\lim_{z\to a} f(z)$ exists in $\mathbb{C}$ and so $\lim_{z \to a}(z-a)f(z) = 0$ (limit of a product).

$\Leftarrow$: If $\lim_{z\to a}(z-a)f(z) = 0$, we can repeat the estimate in the proof of the above Theorem 4.13, with small changes. First fix $\varepsilon > 0$ and choose $r > 0$ small enough that $|z-a| \leq r \Rightarrow |(z-a)f(z)| < \varepsilon$. Then we get,

$$|a_n| \leq \frac{1}{2\pi}(2\pi r) \sup_{|z-a|=r} \frac{|f(z)|}{|z-a|^{n+1}} = r \sup_{|z-a|=r} \frac{|(z-a)f(z)|}{|z-a|^{n+2}} \leq r \frac{\varepsilon}{r^{n+2}} = \frac{\varepsilon}{r^{n+1}}$$

for all sufficiently small $r > 0$. Now if $n < -1$, then $n + 1 > 0$ and so letting $r \to 0^+$ we get $a_n = 0$ ($n \leq -2$). For $n = -1$ we get $|a_{-1}| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this means we must have $a_{-1} = 0$ also. Thus the Laurent series is a power series as before.

Definition 4.15 *An isolated singularity $z = a$ of an analytic function $f(z)$ is called a* pole of f of order p *if the Laurent series for f in a punctured disc about a has the form*

$$f(z) = \frac{a_{-p}}{(z-a)^p} + \frac{a_{-p+1}}{(z-a)^{p-1}} + \cdots = \sum_{n=-p}^{\infty} a_n (z-a)^n$$

with $p > 0$ and $a_{-p} \neq 0$. (This last condition is to ensure that the term with $(z-a)^{-p}$ is really there.)

An isolated singularity $z = a$ of f is called a pole of f *if it is a pole of some order $p > 0$.*

An isolated singularity $z = a$ of f is called an essential singularity of f *if it is neither a pole, nor removable.*

Thus the Laurent series for f in a punctured disc about an essential singularity $z = a$ has the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \quad (0 < |z-a| < \delta)$$

where there are infinitely many $n < 0$ with $a_n \neq 0$. By contrast, for a removable singularity $z = a$ all the negative coefficients vanish ($a_n = 0 \forall n < 0$) and for a pole there is a nonzero finite number of $n < 0$ with $a_n = 0$.

Proposition 4.16 *If an analytic function f has an isolated singularity $z = a$, then it is a pole $\iff$*

$$\lim_{z\to a} |f(z)| = \infty$$

Proof. $\Rightarrow$: If $z = a$ is a pole, then the Laurent series for f in a punctured disk $0 < |z-a| < \delta$ gives

$$f(z) = \sum_{n=-p}^{\infty} a_n (z-a)^n = (z-a)^{-p} \sum_{n=-p}^{\infty} a_n (z-a)^{n+p} = \frac{1}{(z-a)^p} g(z)$$

where $g(z) = \sum_{n=-p}^{\infty} a_n (z-a)^{n+p}$ is analytic for $|z-a| < \delta$, p is the order of the pole and $g(a) = a_{-p} \neq 0$. It follows that

$$\lim_{z\to a} |f(z)| = \lim_{z\to a} \frac{|g(z)|}{|z-a|^p} = \infty$$

$\Leftarrow$: If $\lim_{z\to a} |f(z)| = \infty$, then there exists $\delta > 0$ with $|f(z)| > 1$ for $0 < |z-a| < \delta$. Thus $g(z) = 1/f(z)$ is analytic in the punctured disc $D(a,\delta) \setminus \{0\}$ and also bounded by 1 there ($|g(z)| \leq 1$ for $0 < |z-a| < \delta$). Thus by Theorem 4.13, $g(z)$ can be defined at $z = a$ to make it analytic. In fact $g(a) = \lim_{z\to a} g(z) = \lim_{z\to a} 1/f(z) = 0$. The analytic function g must have a zero of some finite multiplicity $m > 0$ at $z = a$ (by the identity theorem 3.1 applied to $g(z)$ on $D(a,\delta)$. Hence the power series for g about a is of the form

$$g(z) = \sum_{n=m}^{\infty} b_n (z-a)^n = (z-a)^m \sum_{n=m}^{\infty} b_n (z-a)^{n-m} = (z-a)^m h(z)$$

with $h(0) = b_m \neq 0$ and h analytic in a disc about a. Thus there is a $r > 0$ so that $h(z) \neq 0$ for all $z \in D(a,r)$ and $1/h(z)$ is analytic in $D(a,r)$. In $D(a,r)$ we must have a power series expansion

$$\frac{1}{h(z)} = \sum_{n=0}^{\infty} c_n (z-a)^n$$

and so

$$f(z) = \frac{1}{g(z)} = \frac{1}{(z-a)^m h(z)} = \frac{1}{(z-a)^m} \sum_{n=0}^{\infty} c_n (z-a)^n = \sum_{n=0}^{\infty} c_n (z-a)^{n-m}$$

is a Laurent series for f in $0 < |z-a| < r$. Thus f has a pole (of order m) at $z = a$.

Theorem 4.17 (Casorati-Weierstrass) *If an analytic function $f(z)$ has an essential singularity $z = a$, then for all sufficiently small $\delta > 0$ (small enough that $f(z)$ is analytic in the punctured disc $D(a, \delta) \setminus \{a\}$), then*

$$f(D(a, \delta) \setminus \{a\})$$

is dense in $\mathbb{C}$.

Proof. Fix $\delta > 0$ small and put $S = f(D(a, \delta) \setminus \{a\})$. If the closure of S is not all of $\mathbb{C}$, choose $w_0 \in \mathbb{C} \setminus \bar{S}$. As $\mathbb{C} \setminus \bar{S}$ is open there is a disc $D(w_0, \varepsilon) \subset \mathbb{C} \setminus \bar{S}$ with radius $\varepsilon > 0$. Hence, for $z \in D(a, \delta) \setminus \{a\}$ we have $|f(z) - w_0| > \varepsilon$. Thus $g(z) = 1/(f(z) - w_0)$ is analytic in the punctured disc $D(a, \delta) \setminus \{a\}$ and bounded by $1/\varepsilon$ there. Therefore it has a removable singularity at $z = a$ and

$$\lim_{z \to a} g(z) \in \mathbb{C}$$

exists. We can call the limit $g(a)$.

If $g(a) = 0$ then $\lim_{z \to a} |1/g(z)| = \lim_{z \to a} |f(z) - w_0| = \infty$. Hence, as $|f(z| > |f(z) - w_0| - |w_0|$, $\lim_{z \to a} |f(z)| = \infty$. By Proposition 4.16, f must then have a pole at a, a contradiction to the hypotheses.

On the other hand if $g(a) \neq 0$, then

$$\lim_{z \to a} f(z) - w_0 = \frac{1}{g(a)}$$

and so $\lim_{z \to a} f(z) = w_0 + g(a) \in \mathbb{C}$ exists. Thus f has a removable singularity at $z = a$, again a contradiction to the hypotheses.

This S must be dense in $\mathbb{C}$.

Remark 4.18 In an exercise (Exercises 2, question 5) we had

$$f \colon \mathbb{C} \to \mathbb{C} \text{ entire non-constant} \Rightarrow f(\mathbb{C}) \text{ dense in } \mathbb{C}$$

and this was also called the Casorati-Weierstrass theorem. There is a way we can relate the two versions of the theorem.

We say that a function $f(z)$ has an isolated singularity at infinity if $g(\zeta) = f(1/\zeta)$ has an isolated singularity at $\zeta = 0$. That means there is some $R \geq 0$ so that $f(z)$ is analytic for $|z| > R$ and $g(\zeta)$ is analytic for $0 < |\zeta| < 1/R$.

We say that a function $f(z)$ with an isolated singularity at ∞ has a removable singularity at ∞ if $g(\zeta) = f(1/\zeta)$ has a removable singularity at $\zeta = 0$. Similarly, we say that f has a pole at infinity if g has a pole at $\zeta = 0$ and we say f has an essential singularity at ∞ if g has an essential singularity at $\zeta = 0$.

We can apply this terminology to entire functions $f(z)$. Such functions have a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (z \in \mathbb{C})$$

and then

$$g(\zeta) = f(1/\zeta) = \sum_{n=0}^{\infty} a_n \zeta^{-n}$$

is a Laurent series for g valid for $0 < |\zeta|$.

We can see then that g has a removable singularity at 0 if and only if $a_n = 0$ for $n = 1, 2, 3, \ldots$. In other words if and only if $f(z) = a_0$ is constant.

We can see also that g has a pole at $\zeta = 0$ if and only if there are only finitely many n with $a_n \neq 0$, which means that $f(z) = \sum_{n=0}^{N} a_n z^n$ is a polynomial.

Thus the essential singularity case is the case where $f(z)$ is a non-polynomial entire function. If we apply Theorem 4.17 to $g(\zeta) = f(1/\zeta)$ we conclude that, if f is a non-polynomial entire function and $\delta > 0$ then

$$g(D(0,\delta) \setminus \{0\}) \text{ is dense in } \mathbb{C}$$

Hence we have

$$f\colon \mathbb{C} \to \mathbb{C} \text{ entire and not a polynomial } \Rightarrow f(\{z \in \mathbb{C} : |z| > R\}) \text{ dense in } \mathbb{C}$$

for each $R > 0$. (Take $\delta = 1/R$.)

This is a better result than in the exercise, but it does not apply to polynomials. For polynomials we know from the fundamental theorem of algebra that

$$f(z) \text{ a nonconstant polynomial} \Rightarrow f(\mathbb{C}) = \mathbb{C}$$

(because if $w_0 \in \mathbb{C}$ is arbitrary, then the polynomial equation $f(z) = w_0$ has a solution).

In fact all these versions are less than the best result known. Picard's theorem (which we will not prove in this course) states that if f is entire and non-constant, then there is at most one point of $\mathbb{C}$ not in the range $f(\mathbb{C})$. The possibility of an exceptional point is shown by $f(z) = e^z$ which has range $f(\mathbb{C}) = \mathbb{C} \setminus \{0\}$.

There is also a 'Great Picard Theorem' which says that if $f(z)$ is entire and not a polynomial then each equation $f(z) = w_0$ ($w_0 \in \mathbb{C}$) has infinitely many solutions $z \in \mathbb{C}$, except for at most one $w_0 \in \mathbb{C}$. This is often stated: *non-polynomial entire functions take every value in* $\mathbb{C}$ *infinitely often, except for at most one value*. As there can only be finitely many solutions of $f(z) = w_0$ in $|z| \leq R$ (by the identity theorem) it follows that $f(\{z \in \mathbb{C} : |z| > R\})$ contains all $\mathbb{C}$ except at most one point (if f is entire and not a polynomial). This clearly implies the set $f(\{z \in \mathbb{C} : |z| > R\})$ is dense in $\mathbb{C}$. We won't get to the Great Picard theorem either, however.

Definition 4.19 *If $G \subset \mathbb{C}$ is open, then a function $f(z)$ is called* meromorphic on G *if there exists $H \subset G$ open so that $f \colon H \to \mathbb{C}$ is analytic and each point $a \in G \setminus H$ is a pole of f.*

Often this is expressed in the following way: f is meromorphic on G if it is analytic at all points of G except for isolated singularities which are poles.

Lemma 4.20 *If f is meromorphic on $G \subset \mathbb{C}$ open and $K \subset G$ is compact, then there can be at most finitely many poles of f in K.*

Proof. Let $H \subset G$ be the open set where f is actually analytic (with the remaining points of G, those in $G \setminus H$ being all poles).

If there were infinitely many poles of f in K, it would be possible to select an infinite sequence $\alpha_1, \alpha_2, \ldots$ of distinct poles of f inside K. Now, being a sequence in a compact subset of $\mathbb{C}$, $(\alpha_n)_{n=1}^\infty$ must have a convergent subsequence $\left(\alpha_{n_j}\right)_{j=1}^\infty$ and its limit

$$\lim_{j\to\infty} \alpha_{n_j} = \alpha \in K \subset G.$$

Then $\alpha \in H$ or $\alpha \in G \setminus H$.

The case $\alpha \in H$ is not possible since H open would then imply $D(\alpha, r) \subset H$ for some $r > 0$. Thus f analytic on $D(\alpha, r)$ and this implies there are no poles of f in $D(\alpha, r) \Rightarrow \alpha_{n_j} \notin D(\alpha, r) \forall j$. This contradicts α being the limit.

On the other hand the case $\alpha \in G \setminus H$ is also impossible. If $\alpha \in G \setminus H$, then α is an isolated singularity of f and there must be a punctured disc $D(\alpha, r) \setminus \{\alpha\} \subset H$. Now $\lim_{j\to\infty} \alpha_{n_j} = \alpha \Rightarrow \exists j_0$ such that $j \geq j_0$ implies $\alpha_{n_j} \in D(\alpha, r)$. As α_{n_j} is a pole of f, this forces $\alpha_{n_j} = \alpha \forall j \geq j_0$ and contradicts the choice of the α_n as distinct.

Corollary 4.21 If f is meromorphic on $G \subset \mathbb{C}$ open then we can list all the poles of f in a finite or infinite sequence $\zeta_1, \zeta_2, \ldots$.

Proof. If $G = \mathbb{C}$, let $K_n = \overline{D}(0, n)$ and if $G \neq \mathbb{C}$ let

$$K_n = \{z \in G : \text{dist}(z, \mathbb{C} \setminus G) \geq \frac{1}{n} \text{ and } |z| \leq n\}.$$

Here dist$(z, \mathbb{C} \setminus G) = \inf_{w \in \mathbb{C}\setminus G} |z - w|$. Clearly dist$(z, \mathbb{C} \setminus G) \geq 0$ (if $G \neq \mathbb{C}$ and when $G = \mathbb{C}$ we could perhaps interpret it as ∞).

Now K_n is clearly bounded ($K_n \subseteq \overline{D}(0, n)$) and K_n is closed because its complement is

$$\mathbb{C} \setminus K_n = \{w \in \mathbb{C} : |w| > n\} \cup \bigcup_{w \in \mathbb{C}\setminus G} D\left(w, \frac{1}{n}\right)$$

and that is open. Hence K_n is compact for each n.

For $z \in G$ there is some disc $D(z, \delta) \subset G$ and then if $n \in \mathbb{N}$ is large enough that $1/n < \delta$ and $n \geq |z|$ we have $z \in K_n$. Hence $G \subset \bigcup_{n=1}^\infty K_n$. But $K_n \subset G$ for all n and so we also have $\bigcup_{n=1}^\infty K_n \subset G$. Hence

$$\bigcup_{n=1}^\infty K_n = G$$

Now by Lemma 4.20, there can only be a finite number of poles of f in K_1 (or possibly none). We can list the poles in K_1 as a finite list $\zeta_1, \zeta_2, \ldots \zeta_{n_1}$. (Take $n_1 = 0$ if there are no poles in K_1.) Now there are also a finite number of poles in K_2. Let $\zeta_{n_1+1}, \zeta_{n_1+2}, \ldots, \zeta_{n_2}$ be those poles in K_2 but not in K_1. In general let $\zeta_{n_j+1}, \zeta_{n_j+2}, \ldots, \zeta_{n_{j+1}}$ be those poles in K_{j+1} not already in $K_1 \cup K_2 \cup \cdots \cup K_j$.

In this way, we have constructed a complete list $\zeta_1, \zeta_2, \ldots$ of the poles of f in G.

Remark 4.22 We will have further use for these K_n and they have additional useful properties. It is clear from the way they are defined that $K_n \subseteq K_{n+1}$ for each n. In fact K_n is contained in the interior of K_{n+1} because $z \in K_n$ implies

$$D\left(z, \frac{1}{n} - \frac{1}{n+1}\right) \subseteq K_{n+1}$$

It follows then that $G = \bigcup_{n=1}^\infty K_n \subset \bigcup_{n=1}^\infty K_{n+1}^\circ \subset G$ and so

$$G = \bigcup_{n=1}^\infty K_n^\circ.$$

This can be used to show that if $K \subset G$ is compact, then $K \subset K_n^\circ \subset K_n$ for some n.

Any sequence K_n of compact subsets of G with the properties $K_n \subset K_{n+1}^\circ$ and $G = \bigcup_{n=1}^\infty K_n$ is called an *exhaustive sequence* of compact subsets of G. For any exhaustive sequence, we have $K \subset G$ compact $\Rightarrow K \subset K_n$ for some n.

Theorem 4.23 (Identity Theorem for meromorphic functions) *Let $G \subset \mathbb{C}$ be a connected open set and f a meromorphic function on G. If there exists $a \in G$ with $f^{(n)}(a) = 0$ for all $n = 0, 1, 2, \ldots$, then f is identically 0 on G.*

Proof. Let $H \subset G$ be the subset where f is analytic. We cannot immediately apply the identity theorem for analytic functions (3.1) to f on H as we have not assumed H connected.

Now if $z_1, z_2 \in H$, then there is a continuous path [made up of finitely many straight line segments] in G and joining z_1 to z_2 (connected open sets are path connected). The path is a compact subset of G and so passes through at most a finite number of points of $G \setminus H$ (by Lemma 4.20). Around any such point a, there is a punctured disk $D(a, r) \setminus \{a\} \subset H$ and this allows us to divert the path around a. After a finite number of such diversions, we end up with a path in H from z_1 to z_2.

Thus H is path connected and so connected.

Now we can apply the identity theorem to f on H and conclude that $f(z) = 0 \forall z \in H$. This rules out any poles $a \in G \setminus H$ (where $\lim_{z \to a} |f(z)| = \infty$). So $H = G$ and $f \equiv 0$ on G.

Remark 4.24 We can define sums and products of meromorphic functions on $G \subset \mathbb{C}$ open but the definition requires a small bit of care. If f, g are two meromorphic functions on G, then they are analytic on two different open subsets $H_f \subset G$ and $H_g \subset G$.

We define the sum $f+g$ and the product fg on $H_f \cap H_g$ in the 'obvious' way $(fg)(z) = f(z)g(z)$ and $(f+g)(z) = f(z)+g(z)$ on $H_f \cap H_g$. It is not always the case that every point of $G \setminus (H_f \cap H_g) = (G \setminus H_f) \cup (G \setminus H_g)$ is a pole of fg or $f+g$.

For example, if $f(z) = 1/z$ and $g(z) = z$, $G = \mathbb{C}$, $H_f = \mathbb{C} \setminus \{0\}$ and $H_g = \emptyset$. However $f(z)g(z) = 1$ on $H_f \cap H_g$ and has no singularity (r a removable singularity) at $z = 0$.

If we took $f(z) = 1/(z-1) + 1/(z-2)$ and $g(z) = 1/z - 1/(z-1)$, then we find that $H_f = \mathbb{C} \setminus \{1,2\}$, $H_g = \mathbb{C} \setminus \{0,1\}$, $(f+g)(z) = 1/z$ and this has pole at $z = 0$ and $z = 2$, but none at $z = 1$.

In general though all points of $G \setminus (H_f \cap H_g)$ are either poles or removable singularities of fg. So the product makes sense as a meromorphic function. (Similarly for $f+g$.) [**Exercise**: verify.]

Corollary 4.25 *If $G \subset \mathbb{C}$ is connected and $f(z)$ is meromorphic on G but not identically 0, then $1/f$ is meromorphic on G with poles where $f(z) = 0$ if we define $1/f$ to be 0 at poles of f.*

Proof. Let $H \subset G$ be the open subset where f is analytic and $Z_f = \{z \in H : f(z) = 0\}$. Let $P_f = G \setminus H_f$ be the set of poles of f.

Z_f is clearly closed in H and so $H \setminus H_f$ is open. $1/f(z)$ is analytic at all points of $H \setminus H_f$.

If $a \in Z_f$ then there is a disk of positive radius $D(a,r) \subset H$ where f has a power series representation $f(z) = \sum_{n=0}^\infty a_n(z-a)^n$ ($|z-a| < r$). By Theorem 4.23, not all the coefficients a_n can be zero though $f(a) = a_0 = 0$. Thus $f(z) = \sum_{n=m}^\infty a_n(z-a)^n$ with $m \geq 1$ and $a_m \neq 0$.

We can then write $f(z) = (z-a)^m g(z)$ with $g(z) = \sum_{n=m}^\infty a_n(z-a)^{n-m}$ analytic in $D(a,r)$ and $g(a) \neq 0$. So there is some δ with $0 < \delta \leq r$ and $g(z)$ never 0 in $D(a,\delta)$. Hence $Z_f \cap D(a,\delta) = \{a\}$. Also $1/f(z) = (z-a)^{-m}(1/g(z))$ is analytic in the punctured disc $D(a,\delta) \setminus \{a\}$ and has an isolated singularity $z = a$ which is a pole of order m.

At points $b \in P_f$, $\lim_{z\to b} |f(z)| = \infty$ (by Proposition 4.16) and so $\lim_{z \to b} 1/f(z) = 0$. Thus $1/f$ has a removable singularity at $z = b$ (where it should be assigned the value 0).

Also $H \cup P_f$ is open because H is open and $b \in P_f \Rightarrow D(b,\varepsilon) \setminus \{b\} \subset H$ for some $\varepsilon > 0$, which implies $D(b,\varepsilon) \subset H \cup P_f$.

So $1/f$ is now analytic on the open set $H \cup P_f = G \setminus Z_f$ and has isolated singularities at points of Z_f that are all poles.

Remark 4.26 If $G \subset \mathbb{C}$ is open, let $M(G)$ denote all the meromorphic functions on G. We have operations of addition and multiplication on $M(G)$ and since constant functions are in $M(G)$, we can multiply elements of $M(G)$ by complex scalars (same as multiplying by a constant).

These operations make $M(G)$ a *commutative algebra* over $\mathbb{C}$. That means a vector space over $\mathbb{C}$ (addition and multiplication by complex scalars) where multiplication is possible (and certain natural rules are satisfied such as associativity and distributivity). Also the algebra $M(G)$ has a unit element (the constant function 1 has the property that mutliplying it by any $f \in M(G)$ gives f). Commutativity means $fg = gf$.

When G is connected we also have the possibility of dividing by nonzero elements. This makes $M(G)$ a commutative division algebra over $\mathbb{C}$. In particular it is a commutative division ring (forget the vector space structure) and these are called *fields*. As $M(G)$ contains (a copy of) the field $\mathbb{C}$ (in the form of the constant functions), it is an extension field of $\mathbb{C}$.

Theorem 4.27 (Argument principle, simple meromorphic version) *Let $G \subset \mathbb{C}$ be simply connected and f a meromorphic function on G with finitely many zeros $\alpha_1, \alpha_2, \ldots, \alpha_k$ and finitely many poles $\beta_1, \beta_2, \ldots, \beta_\ell$. (We allow $k = 0$ or $\ell = 0$ when there are no zeros or no poles.) Say m_j is the multiplicity of α_j as a zero of f ($1 \leq j \leq k$) and p_j is the order of the pole β_j ($1 \leq j \leq \ell$).*

Let γ be an anticlockwise simple closed curve in $G \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_\ell\}$ (that is in G but not passing through any zeros or poles of f).

Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{k} m_j \mathit{Ind}_\gamma(\alpha_j) - \sum_{j=1}^{\ell} p_j \mathit{Ind}_\gamma(\beta_j)$$

We will often write the right hand side as $N - P$ (or $N_f - P_f$ or $N_{f\gamma} - P_{f,\gamma}$). The first sum counts the number of zeros of f according to multiplicity and the winding number of γ around them and the second sum does a similar thing for poles and their orders.

Proof. The idea is similar to the proof of Theorem 4.8 for the case of analytic functions. We use the residue theorem and show that the residue of f'/f at $z = \alpha_j$ is m_j (the same as before) and the residue of f'/f at $z = \beta_j$ is $-p_j$. Note that f'/f is analytic on G except for isolated singularities at the zeros α_j and poles β_j.

In a punctured disc about a pole $z = \beta_j$, f has a Laurent series

$$\begin{aligned} f(z) &= \sum_{n=-p_j} a_n(z-\beta_j)^n \quad (0 < |z-\beta_j| < r) \\ &= (z-\beta_j)^{-p_j} \sum_{n=-p_j} a_n(z-\beta_j)^{n+p_j} \\ &= (z-\beta_j)^{-p_j} g(z) \end{aligned}$$

Here $g(z)$ is analytic in $|z - \beta_j| < r$ and $g(\beta_j) = a_{-p_j} \neq 0$ and so there exists a positive $\delta \leq r$ so that $g(z)$ is never zero in the disc $D(\beta_j, \delta)$. In the punctured disc $0 < |z - \beta_j| < \delta$ we have

$$\begin{aligned} f'(z) &= -p_j(z-\beta_j)^{-p_j-1}g(z) + (z-\beta_j)^{-p_j}g'(z) \\ \frac{f'(z)}{f(z)} &= \frac{-p_j(z-\beta_j)^{-p_j-1}g(z) + (z-\beta_j)^{-p_j}g'(z)}{(z-\beta_j)^{-p_j}g(z)} \\ &= \frac{1}{z-\beta_j}\frac{-p_j g(z) + (z-\beta_j)g'(z)}{g(z)} \\ &= \frac{1}{z-\beta_j}h(z) \end{aligned}$$

where $h(z)$ is analytic in $D(\beta_j, \delta)$ and $h(\beta_j) = -p_j$. It follows that $h(z)$ has a power series $h(z) = \sum_{n=0}^\infty b_n(z - \beta_j)^n$ in $D(\beta_j, \delta)$ with $b_0 = h(\beta_j) = -p_j$. Thus f'/f has a Laurent series in the punctured disc

$$\frac{f'(z)}{f(z)} = \sum_{n=0}^{\infty} b_n(z-\beta_j)^{n-1} \quad (0 < |z-\beta_j| < \delta)$$

with the coefficient of $(z-\beta_j)^{-1}$ being $b_0 = -p_j$. This the residue of f'/f at $z = \beta_j$ is $-p_j$ as claimed.

Remark 4.28 Instead of assuming that G is simply connected in the theorem above (4.27) we could assume that G is connected and that γ satisfies one of the following restrictions:

(a) γ null homotopic in G

(b) $\mathrm{Ind}_\gamma(w) = 0$ for all $w \in \mathbb{C} \setminus G$

(c) γ a simple closed anticlockwise curve in G with its inside also contained in G. In this case the expression for $N - P$ in the theorem can be simplified because we are in the situation where $\mathrm{Ind}_\gamma(z) = 1$ for z inside γ (and zero for z outside γ). So we get

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{1\le j\le k, \alpha_j \text{ inside } \gamma} m_j - \sum_{1\le j\le \ell, \beta_j \text{ inside } \gamma} p_j$$

To remove the necessity to assume that f has only finitely many zeros and poles we need an improved version of the Residue Theorem.

Theorem 4.29 (Residue theorem, final version) *Let $G \subset \mathbb{C}$ be open and suppose f is analytic in G except for isolated singularities. (That is assume there is $H \subset G$ open so that $f\colon H \to \mathbb{C}$ is analytic and f has an isolated singularity at each point of $G \setminus H$.) Suppose γ is a (piecewise C^1) curve in G that does not pass through any singularity of f (so it is in fact a curve in H) with the property that Ind$_\gamma(w) = 0$ for all $w \in \mathbb{C} \setminus G$.*

Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{a \text{ a singularity of } f} \mathrm{res}(f,a)\mathrm{Ind}_\gamma(a)$$

Though the sum appears potentially infinite we will show that there can be at most finitely many nonzero terms in the sum. Excluding the zero terms we are left with a finite sum and we mean the finite sum.

Proof. To establish first the point about the finiteness of the number of nonzero terms in the sum, let $K = \gamma \cup \{z \in \mathbb{C} : \mathrm{Ind}_\gamma(z) \neq 0\}$. Now $K \subset G$ since $\gamma \subset G$ and $w \in \mathbb{C} \setminus G \Rightarrow \mathrm{Ind}_\gamma(w) = 0 \Rightarrow w \notin K$. (Thus $\mathbb{C} \setminus G \subset \mathbb{C} \setminus K$ or $K \subset G$.)

Also K is compact since it is closed and bounded. K is bounded because Ind_γ is zero on the unbounded component of $\mathbb{C} \setminus \gamma$ (which includes $\{z \in \mathbb{C} : |z| > R\}$ if R is big enough that $\gamma \subset D(0,R)$). K is also closed because Ind_γ is constant on connected components of $\mathbb{C} \setminus \gamma$ which implies that $\mathbb{C} \setminus K = \{z \in \mathbb{C} : \mathrm{Ind}_\gamma(z) = 0\}$ is a union of connected components of $\mathbb{C} \setminus \gamma$ (and these components are all open). Being closed and bounded in $\mathbb{C}$, K is compact.

Now there can only be a finite number of singularities of f in K by an argument similar to the proof in Lemma 4.20 (which was for meromorphic f). [Here is the idea: If there were infinitely many singularities in K we could find an infinite sequence $(a_n)_{n=1}^\infty$ of distinct singularities in K.

Then we could find a subsequence $(a_{n_j})_{j=1}^{\infty}$ converging to a limit $a \in K$. The function f can neither be analytic at a nor have an isolated singularity there.]

We now let H be the open subset of G where f is analytic and $a_1, a_2, \ldots, a_n$ the singularities of f inside K. Let $G_1 = H \cup \{a_1, a_2, \ldots, a_n\}$. Then G_1 is open, $K \subset G_1$ and f is analytic on G_1 except for a finite number of singularities. γ is a curve in $H \subset G_1$ with $\mathrm{Ind}_\gamma(w) = 0$ for all $w \in \mathbb{C} \setminus G_1$ (since such w are not in K). The earlier version of the residue theorem (Theorem 4.3) applies to f on G_1 and implies the result.

Remark 4.30 The argument principle can now be extended to meromorphic functions with potentially infinite numbers of zeros and poles. We need to avoid non-isolated zeros — which would mean f identically zero on some connected component of G by the identity theorem and then the curve γ could not be in such a component because we insist that f is never zero on γ. (We also require that f has no poles on γ).

Since γ has to be in one connected component of G in any case, we can just assume that G is connected, f meromorphic on G and no poles or zeros of f on γ. Then we must make suitable assumptions about γ (such as γ piecewise C^1 closed curve in G with $\mathrm{Ind}_\gamma(w) = 0$ for all $w \in \mathbb{C} \setminus G$).

The argument principle will then state

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{a \in G, f(a)=0} \mathrm{mult}_f(a)\mathrm{Ind}_\gamma(a) - \sum_{b \in G, b \text{ pole of } f} \mathrm{order}_f(b)\mathrm{Ind}_\gamma(b) = N_f - P_f$$

where $\mathrm{mult}_f(a)$ means the multiplicity of a as a zero of f and $\mathrm{order}_f(b)$ means the order of the pole b of f.

Theorem 4.31 (Rouché's theorem) *Suppose f and g are meromorphic functions on a connected open $G \subset \mathbb{C}$ and γ is a piecewise C^1 closed curve in G with*

(a) $\mathrm{Ind}_\gamma(w) = 0$ for all $w \in \mathbb{C} \setminus G$

(b) no zeros or poles of f or g on γ

(c) $|f(z) - g(z)| < |f(z)|$ for all z on γ
[that is, the difference is strictly smaller than one of the functions $|f|$ on γ]

Then

$$N_f - P_f = N_g - P_g$$

where

$$N_f = \sum_{a \in G, f(a)=0} \mathrm{mult}_f(a)\mathrm{Ind}_\gamma(a), \quad P_f = \sum_{b \in G, a \text{ pole of } f} \mathrm{order}_f(b)\mathrm{Ind}_\gamma(b)$$

and similarly for N_g and P_g.

Proof. We could note that the hypotheses (c) actually implies (b) because if there were any poles of either f or g on γ then the inequality (c) would not make sense and if there were zeros of either f or g on γ then the strict inequality could not hold.

Dividing across by $|f(z)|$ we can rewrite (c) as

$$\left|1 - \frac{g(z)}{f(z)}\right| < 1 \iff \frac{g(z)}{f(z)} \in D(1,1)$$

The essence of the proof is that this means $\mathrm{Log}(g(z)/f(z))$ (the principal branch of the $\log$) makes sense for $z \in \gamma$ and gives an antiderivative for $g'(z)/g(z) - f'(z)/f(z)$. To make this antiderivative argument work we need the $\log$ on some open set that contains γ.

Say f is analytic on $H_f \subset G$ open (with poles on $G \setminus H_f$) and g is analytic on $H_g \setminus G$ open (with poles on $G \setminus H_g$). Let $Z_f = \{z \in H_f : f(z) = 0\}$ = the zeros of f. Then $g(z)/f(z)$ is certainly analytic on $(H_f \setminus Z_f) \cap H_g$, an open set that contains γ. By continuity of g/f, the set

$$\{z \in (H_f \setminus Z_f) \cap H_g : g(z)/f(z) \in D(1,1)\}$$

is open and contains γ. On this open set

$$\frac{d}{dz}\mathrm{Log}\,\frac{g(z)}{f(z)} = \frac{1}{\left(\frac{g(z)}{f(z)}\right)}\frac{d}{dz}\left(\frac{g(z)}{f(z)}\right) = \frac{f(z)}{g(z)}\frac{g'(z)f(z) - g(z)f'(z)}{f(z)^2} = \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)}$$

It follows then that the integral of this is zero around the closed curve γ, that is

$$0 = \int_\gamma \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)}\,dz = \int_\gamma \frac{g'(z)}{g(z)}\,dz - \int_\gamma \frac{f'(z)}{f(z)}\,dz$$

Thus

$$\int_\gamma \frac{g'(z)}{g(z)}\,dz = \int_\gamma \frac{f'(z)}{f(z)}\,dz$$

and so by the argument principle

$$2\pi i(N_g - P_g) = 2\pi i(N_f - P_f)$$

and so the result follows.

Example 4.32 (i) We can use Rouché's theorem (4.31) to reprove the fundamental theorem of algebra in yet another way. If $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial of degree $n \geq 1$ (so that $a_n \neq 0$), then the earlier proof started by showing that for $|z| = R > 0$ large enough we have

$$|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| < \frac{1}{2}|a_n||z|^n$$

(We still need this part of the earlier proof and most proofs of the theorem need this part.) If we take $f(z) = a_n z^n$, $g(z) = p(z)$ and γ the circle $|z| = R$ traversed once anticlockwise, then we can apply Rouché's theorem (with $G = \mathbb{C}$ because both functions are entire) since we have

$$|f(z)-g(z)| = |-(a_{n-1}z^{n-1}+\cdots+a_1z+a_0)| < \frac{1}{2}|a_n||z|^n < |a_n z^n| = |f(z)| \text{ for } |z| = R.$$

Thus we conclude

$$N_f - P_f = N_g - P_g$$

or $N_f = N_g$ (since there are no poles in this case). Now $N_f = n$ since $f(z) = 0$ has only the solution $z = 0$ and that has multiplicity n (and $\mathrm{Ind}_\gamma(0) = 1$). So $g(z)$ has $N_g = n$ zeros (counting multiplicities) inside $|z| = R$ (for all large R). It follows that if $n \geq 1$ then $p(z)$ has a zero.

(ii) Show that if $\lambda > 1$ then the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the right half plane $\Re z > 0$.

Solution. This is meant to illustrate the difficulty of applying Rouché's theorem as we have only one function here (need another) and no curve γ. We take $g(z) = \lambda - z - e^{-z}$ the function we want the information about and $f(z) = \lambda - z$ a simpler function to analyse.

We select as our curve γ any closed semicircle of radius $R > \lambda + 1$ in the right half plane, oriented anticlockwise. That is γ is the semicircle $|z| = R$ in $\Re z \geq 0$ plus the segment of the imaginary axis from iR to $-iR$. We take $G = \mathbb{C}$ as our function f and g are entire.

For $z \in \gamma$ we have

$$|f(z) - g(z)| = |e^{-z}| = e^{-\Re z} \leq e^0 = 1.$$

For z on the semicircular part of γ we have

$$|f(z)| = |\lambda - z| \geq |z| - \lambda > R - \lambda > 1 \geq |f(z) - g(z)|.$$

For z on the imaginary axis we have

$$|f(z)| = |\lambda - z| = |\lambda - iy| = \sqrt{\lambda^2 + y^2} \geq \lambda > 1 \geq |f(z) - g(z)|.$$

Thus Rouché's theorem tells us

$$N_f - P_f = N_g - P_g$$

or $N_f = N_g$ since there are no poles. But $N_f = 1$ because $f(z) = 0$ has the solution $z = \lambda$ (which is inside γ and has $\mathrm{Ind}_\gamma(\lambda) = 1$ as γ is simple closed and oriented anticlockwise). Hence $N_g = 1$. Thus $g(z) = \lambda - z - e^{-z} = 0$ has just one solution inside the semicircle as long as $R > \lambda + 1$.

This means there is one in the right half plane (and inside $|z| \leq \lambda + 1$) and there cannot be any other because if there was another solution of $g(z) = 0$ in $\Re z > 0$ we could choose R big enough so that the semicircle would include the second solution (and so would mean $N_g \geq 2$).