Chapter 3: The maximum modulus principle

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Theorem 3.1 (Identity theorem for analytic functions) *Let $G \subset \mathbb{C}$ be open and connected (and nonempty). Let $f: G \to \mathbb{C}$ be analytic. Then the following are equivalent for f:*

(i) $f \equiv 0$

(ii) there is an infinite sequence $(z_n)_{n=1}^{\infty}$ of distinct points of G with $\lim_{n\to\infty} z_n = a \in G$ and $f(z_n) = 0 \forall n$

(iii) there is a point $a \in G$ with $f^{(n)}(a) = 0$ for $n = 0, 1, 2, \ldots$.

Proof. We show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): is really obvious. If $f \equiv 0$, take any $a \in G$ (here we need $G \neq \emptyset$), choose $\delta > 0$ with $D(a, \delta) \subset G$ and put $z_n = a + \delta/(n+1)$.

(ii) $\Rightarrow$ (iii): Assuming $\lim_{n\to\infty} z_n = a \in G$, z_n distinct and $f(z_n) = 0 \forall n$, consider the power series for f centered at a. That is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \text{ for } |z-a| < \delta$$

(for some $\delta > 0$ with $D(a, \delta) \subset G$). Here $a_n = f^{(n)}(a)/n!$ and so our aim of showing $f^{(n)}(a) = 0$ for all $n \geq 0$ is equivalent to showing $a_n = 0$ for all n. If that is not the case, there must be a smallest $m \geq 0$ with $a_m \neq 0$.

Now, for $|z-a| < \delta$ we can write

$$\begin{aligned} f(z) &= \sum_{n=m}^{\infty} a_n (z-a)^n \\ &= (z-a)^m \sum_{n=m}^{\infty} a_n (z-a)^{n-m} \\ &= (z-a)^m g(z) \end{aligned}$$

and $g(z) = \sum_{n=m}^{\infty} a_n(z-a)^{n-m}$ is analytic for $|z-a| < \delta$. Moreover $g(a) = a_m \neq 0$, $g(z)$ is continuous at $z = a$ and so we can find $\delta_0 > 0$, $\delta_0 \leq \delta$, with

$$\begin{aligned} |g(z) - g(a)| &\leq \frac{1}{2}|g(a)| \text{ for } |z-a| < \delta_0 \\ \Rightarrow |g(z)| &\geq \frac{1}{2}|g(a)| \text{ for } |z-a| < \delta_0 \\ \Rightarrow g(z) &\neq 0 \text{ for } |z-a| < \delta_0. \end{aligned}$$

But $0 = f(z_n)$ and for n large enough (say $n > N$) we have $|z_n - a| < \delta_0$ so that $f(z_n) = (z_n - a)^m g(z_n) = 0$. Thus (for $n > N$) $z_n = 0$ or $g(z_n) = 0$. However, we know $g(z_n) \neq 0$ and the z_n are distinct so that at most one n can have $z_n = a$. Hence we are faced with a contradiction.

The contradiction arose from assuming that there was any $a_n \neq 0$. We must therefore have $a_n = 0 \forall n$.

(iii) $\Rightarrow$ (i): Assume now that there is $a \in G$ with $f^{(n)}(a) = 0$ for all $n \geq 0$. Then the power series expansion for f about a (which is valid in a disc $D(a, \delta) \subset G$ with $\delta > 0$) is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n = 0 \text{ for } |z-a| < \delta.$$

Thus $f(z) \equiv 0$ for $|z-a| < \delta$ and differentiating we get $f^{(n)}(z) = 0$ for $n = 0, 1, 2, \ldots$.

This shows that $U = \{a \in G : f^{(n)}(a) = 0 \text{ for all } n = 0, 1, 2, \ldots\}$ is open (and nonempty). U is also closed relative to G. To see that take $b \in G \setminus U$. Then there is some n with $f^{(n)}(b) \neq 0$. Now that $f^{(n)}$ is continuous at b and so there is a $\delta > 0$ so that $f^{(n)}(z) \neq 0$ for all z with $|z - b| < \delta$. This means none of these z can be in U or in other words $D(b, \delta) \subset G \setminus U$. This means $G \setminus U$ is open.

As G is connected, $U \subset G$ nonempty and both open and closed relative to G implies $U = G$. This means $f^{(n)}(z) = 0$ for all $n \geq 0$ and all $z \in G$. Specifically with $n = 0$ we have $f \equiv 0$.

Corollary 3.2 (version with two functions) *Let $G \subset \mathbb{C}$ be open and connected (and nonempty). Let $f, g \colon G \to \mathbb{C}$ be two analytic functions. Then the following are equivalent for f and g:*

(i) $f \equiv g$

(ii) there is an infinite sequence $(z_n)_{n=1}^{\infty}$ of distinct points of G with $\lim_{n\to\infty} z_n = a \in G$ and $f(z_n) = g(z_n) \forall n$

(iii) there is a point $a \in G$ with $f^{(n)}(a) = g^{(n)}(a)$ for $n = 0, 1, 2, \ldots$.

Proof. apply the Identity Theorem 3.1 to the difference $f - g$.

Remark 3.3 The significance of the Identity Theorem is that an analytic function on a connected open $G \subset \mathbb{C}$ is determined on all of G by its behaviour near a single point.

Thus if an analytic function is given on one part of G by a formula like $f(z) = \frac{1}{z-1}$ and that formula makes sense and gives an analytic function on a larger connected subset of G then it has to be that $f(z) = \frac{1}{z-1}$ also holds in the larger set.

This is quite different from what happens with continuous functions like $f\colon \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} z & |z| < 1 \\ \frac{z}{|z|} & |z| \geq 1. \end{cases}$$

Even for C^∞ functions we can have different formulae holding in different places. Consider $g\colon \mathbb{C} \to \mathbb{C}$ where

$$f(z) = \begin{cases} 0 & |z| \leq 1 \\ \exp\left(-\left(\frac{1}{|z|-1}\right)^2\right) & |z| > 1. \end{cases}$$

The original meaning of the word 'analytic' related to this property of analytic functions (one formula).

Corollary 3.4 *If $G \subset \mathbb{C}$ is a connected open set and $f\colon G \to \mathbb{C}$ is analytic and not identically constant, then the* zero set of f

$$Z_f = \{z \in G : f(z) = 0\}$$

has no accumulation points in G.

Proof. First we should define *accumulation point* in case you forget it. If $S \subset \mathbb{C}$ is any set and $a \in \mathbb{C}$, then a is called an accumulation point of S if *for each* $\delta > 0$

$$(S \setminus \{a\}) \cap D(a, \delta) \neq \emptyset.$$

If we a is an accumulation point of S we can choose

$$\begin{aligned} z_1 &\in (S \setminus \{a\}) \cap D(a, 1) \\ z_2 &\in (S \setminus \{a\}) \cap D\left(a, \min\left(\frac{1}{2}, |z_1 - a|\right)\right) \\ z_3 &\in (S \setminus \{a\}) \cap D\left(a, \min\left(\frac{1}{3}, |z_2 - a|\right)\right) \end{aligned}$$

and (inductively) $z_{n+1} \in (S \setminus \{a\}) \cap D\left(a, \min\left(\frac{1}{n}, |z_n - a|\right)\right)$. This produces a sequence $(z_n)_{n=1}^\infty$ of distinct points $z_n \in S$ with $\lim_{n\to\infty} z_n = a$. (It is not hard to see that the existence of such a sequence is equivalent to a being an accumulation point of S.)

Applying this to $S = Z_f$ and using Theorem 3.1 we get $f \equiv 0$.

Corollary 3.5 *Let $G \subset \mathbb{C}$ be open and connected and let $K \subset G$ be compact. Let $f, g\colon G \to \mathbb{C}$ be analytic. If the equation $f(z) = g(z)$ has infinitely many solutions $z \in K$, then $f \equiv g$.*

Proof. Choose an infinite sequence $(z_n)_{n=1}^\infty$ of distinct points $z_n \in K$ where $f(z_n) = g(z_n)$. Since K is compact, the sequence has a convergent subsequence $(z_{n_j})_{j=1}^\infty$ with a limit $a = \lim_{j\to\infty} z_{n_j} \in K \subset G$.

By Corollary 3.2, $f \equiv g$.

Theorem 3.6 (Maximum modulus theorem, basic version) *Let $G \subset \mathbb{C}$ be a connected open set and $f\colon G \to \mathbb{C}$ analytic. If there is any $a \in G$ with $|f(a)| \geq |f(z)|$ for all $z \in G$, then f is constant.*

Proof. (Another way to state this is that $|f(z)|$ cannot have a maximum in G, unless f is constant.)

Choose $\delta > 0$ so that $D(a, \delta) \subset G$. Fix $0 < r < \delta$ and then we have (by the Cauchy integral formula)

$$f(a) = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{f(z)}{z-a}\, dz.$$

Write this out in terms of a parametrisation $z = a + re^{i\theta}$ with $0 \leq \theta \leq 2\pi$, $dz = ire^{i\theta}\, d\theta$.

$$f(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + re^{i\theta})}{ire^{i\theta}}\, d\theta \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta.$$

Hence

$$|f(a)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{i\theta})|\, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a)|\, d\theta = |f(a)|,$$

using $|f(a + re^{i\theta})| \leq |f(a)| \forall \theta$.

We must therefore have equality in the inequalities. Since the integrand $|f(a + re^{i\theta})|$ is a continuous function of θ, this implies $|f(a + re^{i\theta})| = |f(a)|$ for all θ.

Put $\alpha = \text{Arg}(f(a))$. Now

$$\begin{aligned}
|f(a)| &= e^{-i\alpha} f(a) \\
&= \frac{e^{-i\alpha}}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{-i\alpha} f(a + re^{i\theta})\, d\theta \\
\Re|f(a)| = |f(a)| &= \frac{1}{2\pi} \Re \int_0^{2\pi} e^{-i\alpha} f(a + re^{i\theta})\, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \Re(e^{-i\alpha} f(a + re^{i\theta}))\, d\theta \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} |e^{-i\alpha} f(a + re^{i\theta})|\, d\theta \\
&\quad \text{using } \Re w \leq |w| \text{ for } w \in \mathbb{C} \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{i\theta})|\, d\theta \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} |f(a)|\, d\theta = |f(a)|
\end{aligned}$$

Thus we must again have equality in all the inequalities and so

$$\Re(e^{-i\alpha} f(a + re^{i\theta})) = |e^{-i\alpha} f(a + re^{i\theta})| = |f(a)|$$

for all θ. Thus $\Im(e^{-i\alpha}f(a+re^{i\theta})) = 0$ and $e^{-i\alpha}f(a+re^{i\theta}) = |f(a)|$ or $f(a+re^{i\theta}) = e^{i\alpha}|f(a)|$.

Thus $f(z)$ is constant for z in the infinite compact subset $\{z : |z-a| = r\}$ of G. By Corollary 3.5, it follows that $f(z)$ is constant (on G).

Theorem 3.7 (Maximum modulus theorem, usual version) *The absolute value of a nonconstant analytic function on a connected open set $G \subset \mathbb{C}$ cannot have a local maximum point in G.*

Proof. Let $f\colon G \to \mathbb{C}$ be analytic. By a *local maximum point* for $|f|$ we mean a point $a \in G$ where $|f(a)| \geq |f(z)|$ holds for all $z \in D(a,\delta) \cap G$, some $\delta > 0$. As G is open, by making $\delta > 0$ smaller if necessary we can assume $D(a,\delta) \subset G$.

By Theorem 3.6, $|f(a)| \geq |f(z)| \forall z \in D(a,\delta)$ implies f constant on $D(a,\delta)$ (since f must be analytic on $D(a,\delta) \subset G$ and $D(a,\delta)$ is connected open). Then, by the Identity Theorem (Corollary 3.2), f must be constant.

Corollary 3.8 (Maximum modulus theorem, another usual version) *Let $G \subset \mathbb{C}$ be a bounded and connected open set. Let $f\colon \bar{G} \to \mathbb{C}$ be continuous on the closure $\bar{G}$ of G and analytic on G. Then*

$$\sup_{z\in\bar{G}} |f(z)| = \sup_{z\in\partial G} |f(z)|.$$

(That is the maximum modulus of the analytic function $f(z)$ is attained on the boundary ∂G.)

Proof. Since G is bounded, its closure $\bar{G}$ is closed and bounded, hence compact. $|f(z)|$ is a continuous real-valued function on the compact set and so $\sup_{z\in\bar{G}} |f(z)| < \infty$ and the supremum is attained at some point $b \in \bar{G}$. That is $|f(b)| \geq |f(z)| \forall z \in \bar{G}$. If b is on the boundary ∂G then we have

$$|f(b)| = \sup_{z\in\partial G} |f(z)| = \sup_{z\in\bar{G}} |f(z)|$$

but if $b \in G$, then f must be constant by the Identity Theorem 3.1. So in that case $\sup_{z\in\partial G} |f(z)| = \sup_{z\in\bar{G}} |f(z)|$ is also true.

Theorem 3.9 (Fundamental theorem of algebra) *Let $p(z)$ be a nonconstant polynomial ($p(z) = \sum_{k=0}^{n} a_k z^k$ with $a_n \neq 0$ and $n \geq 1$). Then the equation $p(z) = 0$ has a solution $z \in \mathbb{C}$.*

Proof. Dividing the equation by the coefficient of the highest power of z with a nonzero coefficient (a_n in the notation above) we can assume without loss of generality that the polynomial is *monic* (meaning that the coefficient of the highest power of z is 1) and that the polynomial is

$$p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k \qquad (n \geq 1).$$

Now for $|z| = R \geq R_0 = \max\left(1, 2\sum_{k=0}^{n-1}|a_k|\right)$, we have

$$\begin{aligned} |p(z)| &\geq |z|^n - \sum_{k=0}^{n-1}|a_k||z|^k \\ &= R^n - \sum_{k=0}^{n-1}|a_k|R^k \\ &\geq R^n - \sum_{k=0}^{n-1}|a_k|R^{n-1} \\ &\quad \text{using } R \geq 1 \\ &= R^n - R^{n-1}\sum_{k=0}^{n-1}|a_k| \\ &\geq R^n - R^{n-1}(R_0/2) \\ &\geq R^{n-1}(R - R_0/2) \geq R^{n-1}R/2 = R^n/2 = |z|^n/2 \end{aligned}$$

Now if $p(z)$ is never zero for $z \in \mathbb{C}$, then $f(z) = 1/p(z)$ is an entire function. For $|z| \geq R_0$ we have

$$|f(z)| = \frac{1}{|p(z)|} \leq \frac{1}{|z|^n/2} = \frac{2}{|z|^n} \leq \frac{2}{R_0^n}$$

By the maximum modulus theorem (Corollary 3.8)

$$\sup_{|z|\leq R_0}|f(z)| = \sup_{|z|=R_0}|f(z)| \leq \frac{2}{R_0^n}$$

and so we have $|f(z)| \leq 2/R_0^n$ for all $z \in \mathbb{C}$. By Liouville's theorem (1.26) f must be constant. But then $p(z) = 1/f(z)$ is also constant, contradicting our hypotheses.

The contradiction arose by assuming $p(z)$ was never 0. So $p(z) = 0$ for some $z \in \mathbb{C}$.

Remark 3.10 There are many ways to prove the fundamental theorem, but all (or at least most) rely on the same first step as above — for $|z|$ large, $p(z)$ behaves like its term of highest degree $a_n z^n$. To make things a little simpler, we took $a_n = 1$ but that is not really essential. The idea is that, when $|z|$ is large the highest term outweighs the combination of the lower degree terms.

Also, although we used the maximum modulus theorem to get the same bound for $f(z)$ for $|z| \leq R_0$ as for $|z| \geq R_0$, we could have just used compactness to get $\sup_{|z|\leq R_0}|f(z)| < \infty$. (We used an argument like that once before in the proof of the winding number version of Cauchy's theorem (1.30 claim 2 in the proof).

Corollary 3.11 *If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n (meaning $a_n \neq 0$) then $p(z)$ can be factored*

$$p(z) = a_n \prod_{j=1}^{n}(z - \alpha_j)$$

for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$.

Proof. For $n = 0$ the result is true (provided we interpret the empty product as 1).

By the Fundamental Theorem of algebra 3.9, there is some $\alpha_1 \in \mathbb{C}$ with $p(\alpha_1) = 0$. Then it follows from the remainder theorem that $z - \alpha_1$ divides $p(z)$. In other words $p(z) = (z - \alpha_1)q(z)$ where $q(z)$ has degree $n - 1$ and leading coefficient a_n (same as for $p(z)$).

If we arrange the proof more formally as induction on the degree, we have the starting $n = 0$ and the induction step. Applying the result from degree $n - 1$ to $q(z) = a_n \prod_{j=2}^{n}(z - \alpha_j)$ we are done.