Chapter 2: Logarithms and simple connectedness

Course 414, 2003–04

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Definition 2.1 Let $G \subseteq \mathbb{C}$ be connected. Then G is called simply connected if every closed curve γ in G is null homotopic in G.

Example 2.2 Every convex set G (for example $G = \mathbb{C}$ or $G =$ a disc) is simply connected.

Notice that the definition of simple connectivity is a purely topological definition — there is no analytic function theory or even C^1 curves involved.

Theorem 2.3 *Let $f\colon G \to \mathbb{C}$ be analytic on a simply connected open $G \subset \mathbb{C}$. Then*

$$\int_\gamma f(z)\,dz = 0$$

for every (piecewise C^1) closed curve γ in G.

Proof. Since γ is null-homotopic in G, this follows by the homotopy version of Cauchy's theorem (Corollary 1.43).

Theorem 2.4 *Let $f\colon G \to \mathbb{C}$ be analytic on a connected open set $G \subset \mathbb{C}$. Then f has the property that $\int_\gamma f(z)\,dz = 0$ holds for each (piecewise C^1) closed curve γ in $G \iff f$ has an antiderivative in G.*

Antiderivative is a function $F\colon G \to \mathbb{C}$ with $F' = f$.

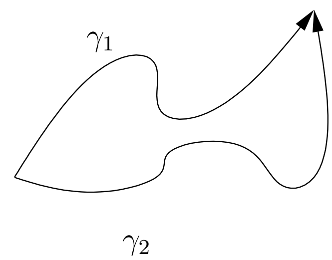
Proof. $\Leftarrow$: If $\gamma\colon [a,b] \to G$ is a (piecewise C^1) closed curve in G and F is an antiderivative of f, then

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)) = 0.$$

$\Rightarrow$: If γ_1 and γ_2 are two piecewise C^1 curves in G with the same starting point and the same ending point, then

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

because γ_1 followed by γ_2 reversed makes a closed curve in G and so the integral of f around this closed curve is 0 by our hypothesis.



Now fix $a \in G$.

Define $F\colon G \to \mathbb{C}$ by $F(z) = \int_a^z f(\zeta)\, d\zeta$. Here we use the fact that G connected open implies G path connected and so there exists a piecewise C^1 curve in G from a to z. Moreover the integral from a to z does not depend on what path is chosen by the remark above.

Now we can show $F'(z) = f(z) \forall z \in G$ in the same way as in the proof of Cauchy's theorem for a convex set (1.16).

Corollary 2.5 *If $G \subset \mathbb{C}$ is a simply connected open set, then every analytic $f\colon G \to \mathbb{C}$ has an antiderivative in G.*

Proof. Combine last two results.

Remark 2.6 Much later we will show that the converse is true: if $G \subset \mathbb{C}$ is a connected open set with the property that each analytic $f\colon G \to \mathbb{C}$ has an antiderivative, then G must be simply connected.

Our proof will rely on the Riemann mapping theorem.

Note 2.7 Recall now the exponential function $\exp z = \sum_{n=0}^{\infty} z^n/n!$. Also $e = \exp(1)$, $e^z = \exp(z)$, $e^{z+w} e^z e^w$, $e^0 = 1$, $e^{-z} = 1/e^z$, $\frac{d}{dz} e^z = e^z$, $e^z = 1 \iff z = 2n\pi i$ for some $n \in \mathbb{Z}$, $|e^{ix}| = 1 \forall x \in \mathbb{R}$.

Definition 2.8 Let $f\colon G \to \mathbb{C}$ be an analytic function on an open set $G \subset \mathbb{C}$. Then an analytic function $g\colon G \to \mathbb{C}$ is a *branch of the logarithm of f on G* if $e^{g(z)} \equiv f(z)$.

Proposition 2.9 *Let $f\colon G \to \mathbb{C}$ be analytic on $G \subset \mathbb{C}$ open. Then*

(i) if there is a branch of the logarithm of f on G, then $f(z)$ is never 0 on G (that is $\forall z \in G, f(z) \neq 0$);

(ii) if g is a branch of the logarithm of f on G, then $\Re g(z) = \log |f(z)|$ ($\forall z \in G$);

(iii) if g_1 and g_2 are two branches of the logarithm of f on G, and if G is connected, then $g_1(z) - g_2(z) \equiv 2n\pi i$ for some $n \in \mathbb{Z}$ (constant for $z \in G$);

(iv) if G is connected then an analytic $g\colon G \to \mathbb{C}$ is a branch of the logarithm of f on G if and only if g has the following two properties:

(a)

$$g'(z) = \frac{f'(z)}{f(z)}$$

(that is g is an antiderivative of f'/f)

(b) there exists one $z_0 \in G$ with $\exp(g(z_0)) = f(z_0)$.

(To be 100% precise we need to assume G is not empty here.)

Proof.

(i) is true because

$$e^{g(z)}e^{-g(z)} = e^0 = 1 \Rightarrow f(z)e^{-g(z)} = 1 \Rightarrow f(z) \neq 0.$$

(ii)

$$\begin{aligned} e^{g(z)} = f(z) \quad &\Rightarrow \quad e^{\Re g(z)}e^{i\Im g(z)} = f(z) \\ &\Rightarrow \quad \left|e^{\Re g(z)}\right|\left|e^{i\Im g(z)}\right| = |f(z)| \\ &\Rightarrow \quad e^{\Re g(z)} = |f(z)| \end{aligned}$$

and hence $\Re g(z)$ is the ordinary real logarithm $\log|f(z)|$.

(iii) We have

$$e^{g_1(z)-g_2(z)} = \frac{e^{g_1(z)}}{e^{g_2(z)}} = \frac{f(z)}{f(z)} = 1.$$

Hence, for each $z \in G$ $g_1(z) - g_2(z) = 2n\pi i$ for some $n = n(z) \in \mathbb{Z}$. But $z \mapsto n(z) = (g_1(z) - g_2(z))/(2\pi i)$ is a continuous integer-valued function on a connected G. Hence it must be constant.

(iv) $\Rightarrow$: Differentiating $e^{g(z)} = f(z)$ results in $e^{g(z)}g'(z) = f'(z)$ and so $f(z)g'(z) = f'(z)$ or $g'(z) = f'(z)/f(z)$.

$\Leftarrow$: Suppose $g'(z) = f'(z)/f(z)$. Differentiate $e^{-g(z)}f(z)$ to get

$$\frac{d}{dz}e^{-g(z)}f(z) = e^{-g(z)}(-g'(z))f(z) + e^{-g(z)}f'(z) = -e^{-g(z)}f'(z) + e^{-g(z)}f'(z) = 0.$$

Hence (since G connected) $e^{-g(z)}f(z) = C =$ constant in G. Hence $f(z) = Ce^{g(z)}$. Plugging in $z = z_0$ shows $C = 1$ and so $f(z) = e^{g(z)}$. This g is a branch of the logarithm of f on G.

We often write "g is a branch of $\log f$ (on G)" for this even though it is not always the case that there is a branch of $\log f$.

Example 2.10 There is no branch of $\log z$ in $\mathbb{C} \setminus 0$.
Proof. Such a branch would be an antiderivative of

$$\frac{\frac{d}{dz}z}{z} = \frac{1}{z}$$

on $\mathbb{C} \setminus 0$. If there were such an antiderivative then we would have

$$\int_{|z|=1} \frac{1}{z}\,dz = 0$$

and this is **not so**. Hence there can be **no branch** of $\log z$ on $\mathbb{C} \setminus 0$.

Proposition 2.11 *If $G \subset \mathbb{C}$ is open an $f\colon G \to \mathbb{C}$ is analytic and $g\colon G \to \mathbb{C}$ is a continuous function with $e^{g(z)} \equiv f(z)$, then g is a branch of the logarithm of f on G.*

(The point here is that we do not need to assume that g is analytic. If g is continuous (and f is analytic) then g is automatically analytic.)

Proof. For the proof it is sufficient to deal with the case where $G = D(a, r)$ is a a disc, because analyticity of g is a local property about g. [If we show g is analytic in a disc about each $a \in G$, then we know $g'(a)$ exists for each $a \in G$.]

If $e^{g(z)} = f(z)$ then certainly $f(z)$ is never zero and we can find an antiderivative $h(z)$ for $f'(z)/f(z)$ on $D(a, r)$ (by Corollary 2.5). If we take $h_1(z) = h(z)-h(a)+g(a)$ we have another antiderivative ($h_1'(z) = h'(z) = f'(z)/f(z)$) and this one has $e^{h_1(a)} = e^{g(a)} = f(a)$. Thus h_1 is a branch of $\log f$ on $D(a, r)$ by Proposition 2.9(iv). Hence

$$e^{g(z)-h_1(z)} = \frac{e^{g(z)}}{e^{h_1(z)}} = \frac{f(z)}{f(z)} = 1$$

and so $g(z) - h_1(z) = 2n(z)\pi i$ for some $n(z) \in \mathbb{Z}$. But then $n(z) = (g(z) - h_1(z))/(1\pi i)$ is an integer-valued continuous function on the connected $D(a, r)$. Hence $n(z) = n(a) =$ constant and $g(z) = h_1(z) + 2n(a)\pi i$ is analytic.

Theorem 2.12 *Let $G \subset \mathbb{C}$ be a connected open set. Then the following are equivalent properties for G to have:*

(i) If $f\colon G \to \mathbb{C}$ is analytic and γ is a piecewise C^1 closed curve in G, then $\int_\gamma f(z)\,dz = 0$.

(ii) Every analytic function $f\colon G \to \mathbb{C}$ has an antiderivative.

(iii) If $f\colon G \to \mathbb{C}$ is analytic and never zero in G, then there is a branch of the logarithm of f on G.

(iv) For every $w \in \mathbb{C} \setminus G$ and every piecewise C^1 closed curve γ in G, $\mathrm{Ind}_\gamma(w) = 0$.

(v) Every nowhere-vanishing analytic function $f\colon G \to \mathbb{C}$ has an analytic m^{th} root for each $m = 2, 3, \ldots$ (that is an analytic $g_m\colon G \to \mathbb{C}$ with $(g_m(z))^m \equiv f(z)$).

(vi) Every nowhere-vanishing analytic function $f: G \to \mathbb{C}$ has an analytic square root.

Proof. We will show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) and so establish that if any one of these four properties holds for G, then each of the others must hold. So all four will be shown equivalent.

Then we will show (iii) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (iv), showing that each of the last two are equivalent to the first four.

(i) $\Rightarrow$ (ii): by Theorem 2.4

(ii) $\Rightarrow$ (iii): Assume (ii) holds and we have a nowhere-vanishing analytic $f: G \to \mathbb{C}$, then we know there is an antiderivative $g: G \to \mathbb{C}$ for $f'(z)/f(z)$. Differentiating $e^{-g(z)}f(z)$ we get

$$-g'(z)e^{-g(z)}f(z) + e^{-g(z)}f'(z) = -\frac{f'(z)}{f(z)}e^{-g(z)}f(z) + e^{-g(z)}f'(z) = 0$$

and hence $e^{-g(z)}f(z) = c$ is a constant. The constant $c \neq 0$ and so there is a $w \in \mathbb{C}$ with $e^w = z$. If we fix $z_0 \in G$ and take $g_1(z) = g(z) - g(z_0) + w$, then we have $g_1'(z) = f'(z)/f(z)$ and $\exp(g_1(z_0)) = f(z_0)$. By Proposition 2.9(iv), g_1 is a branch of $\log f$ on G.

(iii) $\Rightarrow$ (iv): Assume now (iii) holds and $w \in \mathbb{C} \setminus G$. Take $f(z) = z - w$ (analytic and nowhere zero on G) and g to be a branch of $\log f(z) = \log(z - w)$ on G. Then by Proposition 2.9, we have $g'(z) = f'(z)/f(z) = 1/(z - w)$. So

$$\mathrm{Ind}_\gamma(w) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - w}\, dz = 0$$

for any piecewise C^1 closed curve γ in G (by 1.12).

(iv) $\Rightarrow$ (i): holds by the winding number version of Cauchy's theorem (1.30).

(iii) $\Rightarrow$ (v): If $e^{g(z)} = f(z)$ with g analytic, take $g_m(z) = \exp(g(z)/m)$. Then g_m is analytic and $(g_m(z))^m = (\exp(g(z)/m))^m = \exp(g(z)) = f(z)$.

(v) $\Rightarrow$ (vi): is immediate by taking $m = 2$.

(vi) $\Rightarrow$ (iv): Assume now that (vi) holds. Fix $w \in \mathbb{C} \setminus G$ and a piecewise C^1 closed curve $\gamma: [a, b] \to G$.

Let $f_0: G \to \mathbb{C}$ be $f_0(z) = z - w$, which is analytic and nowhere-vanishing on G. Thus there exists an analytic square root $f_1: G \to \mathbb{C}$ ($(f_1(z))^2 \equiv f_0(z) = z - w$). Since $f_1(z)$ is never zero, it has a square root f_2 analytic on G (with $(f_2(z))^2 = f_1(z) \forall z \in G$). We can continue to extract square roots and (by induction on n) show there is a sequence of analytic functions $f_n: G \to \mathbb{C}$ ($n = 1, 2, \ldots$) with $(f_{n+1}(z))^2 = f_n(z) \forall z \in G$, $n = 0, 1, 2, \ldots$.

Consider the curves $\gamma_n: [a, b] \to \mathbb{C}$ given by $\gamma_n = f_n \circ \gamma$ (for $n = 0, 1, 2, \ldots$). Then we have

$$\begin{aligned}
\mathrm{Ind}_{\gamma_0}(0) &= \frac{1}{2\pi i}\int_{\gamma_0} \frac{1}{z}\, dz \\
&= \frac{1}{2\pi i}\int_a^b \frac{1}{f_0(\gamma(t))} f_0'(\gamma(t))\gamma'(t)\, dt \\
&= \frac{1}{2\pi i}\int_a^b \frac{1}{\gamma(t) - w}\gamma'(t)\, dt \\
&= \frac{1}{2\pi i}\int_\gamma \frac{1}{z - w}\, dz = \mathrm{Ind}_\gamma(w)
\end{aligned}$$

Also $f_{n+1}^2 = f_n$ implies $2f_{n+1}(z)f'_{n+1}(z) = f'_n(z)$ (by differentiation) and so we can say

$$\begin{aligned}
\mathrm{Ind}_{\gamma_n}(0) &= \frac{1}{2\pi i}\int_{\gamma_n}\frac{1}{z}\,dz \\
&= \frac{1}{2\pi i}\int_a^b \frac{1}{f_n(\gamma(t))} f'_n(\gamma(t))\gamma'(t)\,dt \\
&= \frac{1}{2\pi i}\int_a^b \frac{1}{(f_{n+1}(\gamma(t)))^2} 2f_{n+1}(\gamma(t))f'_{n+1}(\gamma(t))\gamma'(t)\,dt \\
&= 2\frac{1}{2\pi i}\int_a^b \frac{1}{f_{n+1}(\gamma(t))} f'_{n+1}(\gamma(t))\gamma'(t)\,dt \\
&= 2\frac{1}{2\pi i}\int_{\gamma_{n+1}}\frac{1}{z}\,dz \\
&= 2\mathrm{Ind}_{\gamma_{n+1}}(0)
\end{aligned}$$

is divisible by 2 (because the indices are integers). This shows that

$$\mathrm{Ind}_\gamma(w) = \mathrm{Ind}_{\gamma_0}(0) = 2\mathrm{Ind}_{\gamma_1}(0) = 2^n\mathrm{Ind}_{\gamma_n}(0)$$

is an integer divisible by 2^n for every $n = 1, 2, \ldots$. This can only happen if $\mathrm{Ind}_\gamma(w) = 0$.

Remark 2.13 We know (Theorem 2.3) that if G is simply connected and open in $\mathbb{C}$, then it satisfies 2.12 (i) and hence all the other equivalent conditions.

Intuitively 2.12 (iv) says that G has “no holes” (around which one can place a simple closed curve in G) and it is tempting to believe that all the properties of Theorem 2.12 are equivalent to G being simply connected. This is in fact true, but we cannot prove it now.

Here is another condition which will also turn out to be equivalent to G simply connected.

Proposition 2.14 *If $G \subseteq \mathbb{C}$ is a connected open set with the property that each connected component of $\mathbb{C} \setminus G$ is unbounded, then G satisfies the equivalent conditions of 2.12.*

Proof. We show that G must satisfy 2.12 (iv). If γ is a piecewise C^1 closed curve in G, then $\mathbb{C}\setminus G \subset \mathbb{C}\setminus\gamma$. Hence each connected component of $\mathbb{C}\setminus G$ is contained in a connected component of $\mathbb{C}\setminus\gamma$. All connected components of $\mathbb{C}\setminus G$ are unbounded, whereas $\mathbb{C}\setminus\gamma$ has just one unbounded component by compactness of γ. Hence if $w \in \mathbb{C}\setminus\gamma$, then $\mathrm{Ind}_\gamma(w) = 0$ because w is in the unbounded component of $\mathbb{C}\setminus\gamma$.

Example 2.15 Returning to the example of $\log z$, let $G = \mathbb{C}\setminus(-\infty, 0]$ be the complement in $\mathbb{C}$ of the negative real axis (or to be more precise the complement of the non-positive real axis). Then G i clearly connected and its complement has just one connected component, the real interval $(-\infty, 0]$ (which is of course unbounded).

By Proposition 2.14, since $f(z) = z$ is analytic and never zero on G we can say that there is a $g\colon G \to \mathbb{C}$ analytic which is a branch of $\log f(z) = \log z$ on G. That is $\exp(g(z)) \equiv z$.

Since $\exp(g(1)) = 1$, we have $g(1) = 2n\pi i$ for some $n \in \mathbb{Z}$ and if we replace $g(z)$ by $h(z) = g(z) - 2n\pi i$ we get a branch of $\log z$ in G with $h(1) = 0$.

There can only be one such branch. (See Proposition 2.9 (iii).)

This branch of $\log z$ is called the *principal branch* and sometimes denoted $\operatorname{Log} z$. It has the properties

$$\begin{aligned} e^{\operatorname{Log} z} &= z \text{ in } \mathbb{C} \setminus (-\infty, 0] \\ \operatorname{Log} 1 &= 0 \\ \operatorname{Log} z & \quad \text{analytic in } \mathbb{C} \setminus (-\infty, 0] \end{aligned}$$

We also know $(d/dz) \operatorname{Log} z = 1/z$ in $\mathbb{C} \setminus (-\infty, 0]$.

Now $\Re(\operatorname{Log} z) = \log |z|$ and if we write (just for a moment) $\phi = \Im(\operatorname{Log} z)$, then we have

$$\begin{aligned} e^{\log |z| + i\phi} &= z \\ e^{\log |z|} e^{i\phi} &= z \\ |z| e^{i\phi} &= z \\ |z|(\cos \phi + i \sin \phi) &= z \end{aligned}$$

so that ϕ is a value for the argument of z (more usually denoted θ). We see that a choice of a branch of $\log z$ is a choice of $\arg z$ so that $\log z = \log |z| + i \arg z$ is analytic. (From Proposition 2.11 we know that choosing the argument continuously is the key thing — analyticity will automatically follow).

The principal branch $\operatorname{Log} z$ chooses the argument θ in the range $-\pi < \theta < \pi$ (which is also known as the *principal branch* of the argument of z).

If $G \subset \mathbb{C} \setminus \{0\}$ is any open connected set where each component of $\mathbb{C} \setminus G$ is unbounded, then we know from Proposition 2.14 that there has to be a branch of $\log z$ on G. However the principal branch $\operatorname{Log} z$ will not work on G unless G does not meet the negative real axis.

If G is connected open and the complement of G is an injective continuous curve $\sigma \colon [0, 1) \to \mathbb{C}$ where $\sigma(0) = 0$ and $\lim_{t \to 1^-} |\sigma(t)| = \infty$, then the complement of G is the unbounded connected set $\sigma([0, 1))$. So we can find a branch of $\log z$ in G. By taking σ to be a spiral we can find examples where the imaginary part of $\log z$ will be unbounded on G (for all branches of $\log z$ on G).

Here are some of the elementary properties of $\operatorname{Log} z$.

Proposition 2.16 *Let $G = \mathbb{C} \setminus [0, -\infty)$.*

(i) For $z, w \in G$ such that $zw \in G$, $\operatorname{Log}(zw) = \operatorname{Log} z + \operatorname{Log} w + 2n\pi i$ for some $n \in \mathbb{Z}$.

(ii) If $z \in G$ and $k \in \mathbb{Z}$, then

$$z^k = e^{k \operatorname{Log} z}$$

(iii) If $z \in G$, $k \in \mathbb{Z}$ and $z^k \in G$, then $\operatorname{Log}(z^k) = k \operatorname{Log} z + 2n\pi i$ (for some $n \in \mathbb{Z}$).

Proof. These are all quite easy to check. For example,

$$\exp(\operatorname{Log}(zw)) = zw = \exp(\operatorname{Log} z)\exp(\operatorname{Log} w) = \exp(\operatorname{Log} z + \operatorname{Log} w)$$

implies that $\operatorname{Log}(zw)$ and $\operatorname{Log} z + \operatorname{Log} w$ differ by a multiple of $2\pi i$.

Definition 2.17 If $w \in \mathbb{C}$ is arbitrary and $z \in \mathbb{C} \setminus [0, -\infty)$, then we define the *principal branch* (or principal value) of z^w by

$$z^w = (e^{\operatorname{Log} z})^w = e^{w \operatorname{Log} z}.$$

[Notice that there could be a case for other values $e^{w(\operatorname{Log} z + 2n\pi i)}$ with $n \in \mathbb{Z}$. If w is a rational value (say p/q), then there are only finitely many different possible values, but for irrational or complex w we would have infinitely many different possible values we could plausibly attach to z^w via logs.]

Proposition 2.18 *(i) For $w \in \mathbb{C}$ fixed and $f\colon \mathbb{C} \setminus [0, -\infty) \to \mathbb{C}$ given by $f(z) = z^w$ (principal value), then $f'(z) = wz^{w-1}$.*

(ii) For $a \in \mathbb{C} \setminus [0, -\infty)$ fixed and $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(z) = a^z$ (again (principal value), then $f'(z) = (\operatorname{Log} a)a^z$.

Proof. Simple to show just using the definitions and the chain rule.