Chapter 1: Cauchy's theorem, various versions

Course 414, 2003–04

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Definition 1.1 If $G \subset \mathbb{C}$ is open, $f: G \to \mathbb{C}$ a function and $z_0 \in G$, then f is *differentiable* at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The value of the limit is denoted by $f'(z_0)$ when it exists.

If $f'(z_0)$ exists for all $z_0 \in G$, then we say that f is *analytic* on G. (The terms *holomorphic* or *regular* are also used.)

Remarks 1.2 (a) $f'(z_0)$ exists $\Rightarrow f$ is continuous at z_0.

(b) $f: G \to \mathbb{C}$ analytic on $G \Rightarrow f$ is continuous on G.

(c) Sums, scalar multiples and products of analytic functions are analytic. Also quotients on a domain where the denominator is never zero.

(d) $f(z) = a_n z^n + \cdots a_1 z + a_0 \Rightarrow f'(z) = n a_n z^{n-1} + \cdots 2a_2 z + a_1$.

(e) The chain rule holds: $(f \circ g)'(z) = f'(g(z))g'(z)$

Terminology 1.3 A *region* in $\mathbb{C}$ is a connected open subset $G \subset \mathbb{C}$.

If an open $G \subseteq \mathbb{C}$ has more than one connected component, then analytic functions $f: G \to \mathbb{C}$ can be made by arbitrarily specifying analytic functions $f_i: G_i \to \mathbb{C}$ on each connected component G_i of G and then taking $f(z)$ so that $f(z) = f_i(z)$ for $z \in G_i$.

The different f_i need not be related to one another in any way.

For this reason it is rarely necessary to consider analytic functions on disconnected G, though some theorems remain true even for disconnected G if they have hypotheses and conclusions that are 'local'. Being analytic is a local condition (only needs to be checked by working near any given point), but to say that f is constant is a global condition on f on all of its domain.

For example, the theorem that says

$$f'(z) \equiv 0 \Rightarrow f \text{ constant}$$

is true only under the assumption that the domain of f is connected. You can make a version of the theorem which is true for disconnected domains and says

> $f: G \to \mathbb{C}$ analytic and $f'(z) \equiv 0$ implies f is constant on each connected component of G.

However, the proof is just to work on each connected component independently of the others.

Theorem 1.4 (Cauchy-Riemann equations) *Suppose $G \subset \mathbb{C}$ is open and $f: G \to \mathbb{C}$ is a function. Write $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ where $u(x,y) = \Re f(x+iy)$ is the real part and $u(x,y) = \Im f(x+iy)$ the imaginary part.*

(i) If f is analytic, then u, v satisfy

$$\left\{ \begin{array}{rcl} \dfrac{\partial u}{\partial x} &=& \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} &=& -\dfrac{\partial u}{\partial y} \end{array} \right. \qquad (\text{C-R equations})$$

(ii) If u and v are C^1 functions on G (regarded as a domain in $\mathbb{R}^2$) which satisfy the C-R equations, then f is analytic on G.

Remarks 1.5 The two parts of 1.4 are almost, but not quite converses, because the additional assumption is made in the second part that u and v are C^1 functions (continuous first order partials). This assumption makes the proof relatively simple — it comes down to knowing from real analysis that u and v must have derivatives (or total derivatives), which is defined to mean that the linear approximation formula works. As a map from a domain $G \subset \mathbb{R}^2$ with values in $\mathbb{R}^2$, f is C^1 and must have a derivative too, given by the linear map

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \mapsto \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

(where the partials are evaluated at the point in question). The C-R equation are equivalent to saying that this linear map $: \mathbb{R}^2 \to \mathbb{R}^2$ is actually a complex linear map

$$h_i + ih_2 \mapsto \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) (h_i + ih_2)$$

and this is the essence of the proof of the second part of 1.4.

We could enhance the first part to add the fact that u and v must in fact be C^1 (using Goursat's theorem below, this is true) and this would make the two parts converses of one another.

It is in fact true that the second part is true without assuming that u and v are C^1. But this requires some difficult results from partial differential equations, which say that solutions (even generalised solutions) of the C-R equations are automatically C^1. We will not be able to prove this fact in this course.

The C-R equations (two real equations) can be written as a single complex equation by introducing the so call D-bar operator $\bar{\partial}$. For $f = u + iv$ as above, we define

$$\bar{\partial} f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

where by $\partial f/\partial x$ and $\partial f/\partial y$ we mean

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial x} \\ \frac{\partial f}{\partial y} &= \frac{\partial u}{\partial y} + i\frac{\partial u}{\partial y} \end{aligned}$$

Then the single equation

$$\bar{\partial} f = 0 \qquad (\bar{\partial} \text{ equation})$$

is equivalent to the system of C-R equations.

Definition 1.6 A *power series* centered at $a \in \mathbb{C}$ is a series

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

Recall 1.7 Every power series has a radius of convergence R which is a "number" satisfying

1. $0 \leq R \leq \infty$
2. The power series converges (absolutely) for all z with $|z - a| < R$
3. The power series diverges for all z with $|z - a| > R$

Moreover

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

The existence of R can be shown by using the n^{th} root test for convergence of a series, which states

> For a series $\sum_{n=0}^{\infty} z_n$, consider $\rho = \limsup_{n\to\infty} |z_n|^{1/n}$. If $\rho < 1$, the series converges (absolutely) but if $\rho > 1$ the series diverges.

For a power series centered at a and with radius of convergence R, the disk $D(a, R)$ (or its boundary circle) is called the circle of convergence. The power series converges inside the (interior of) its circle of convergence, but diverges outside the circle.

An important fact is that a power series as above converges *uniformly* in any strictly smaller circle $\bar{D}(a, r) \subset D(a, R)$ with $0 \leq r < R$. This can be checked by using the Weierstrass M-test for uniform convergence of a series of functions:

For a series $\sum_{n=0}^\infty f_n(z)$ of functions $f_n \colon S \to \mathbb{C}$ all defined on some set S, if there exists a convergent series $\sum_{n=0}^\infty M_n$ (of constants) such that $|f_n(z)| \leq M_n$ holds for all n and all $z \in S$, then the series $\sum_{n=0}^\infty f_n(z)$ converges uniformly on S.

Theorem 1.8 *If $\sum_{n=0}^\infty a_n(z-a)^n$ is a power series with radius of convergence $R > 0$ then*

$$f(z) = \sum_{n=0}^\infty a_n(z-a)^n \qquad (z \in D(a,R))$$

defines an analytic function $f \colon D(a,R) \to \mathbb{C}$ and its derivative is given by

$$f'(z) = \sum_{n=1}^\infty n a_n (z-a)^{n-1}$$

This latter series for $f'(z)$ has the same radius of convergence R (and consequently $f'(z)$ is analytic in $D(a,R)$).

It follows that $f^{(2)}(z) = f''(z)$, $f^{(3)}(z) = f'''(z)$, ... are all analytic in $D(a,R)$ and are each representable by power series in $D(a,R)$.

Proof. First the series $\sum_{n=1}^\infty n a_n (z-a)^{n-1}$ has radius of convergence

$$\frac{1}{\limsup_{n\to\infty} |n a_n|^{1/(n-1)}} = \frac{1}{\limsup_{n\to\infty} n^{1/(n-1)} \left(|a_n|^{1/n}\right)^{n/(n-1)}}$$

and we can show that this is R using $\lim_{n\to\infty} n^{1/(n-1)} = 1$.

Fix z with $|z-a| < R$ and let $g(z) = \sum_{n=1}^\infty n a_n (z-a)^{n-1}$ (which we now know converges). Consider (for $h \neq 0$)

$$\begin{aligned} \left| \frac{f(z+h) - f(z) - g(z)h}{h} \right| &= \left| \sum_{n=0}^\infty a_n \left((z+h-a)^n - (z-a)^n - n(z-a)^{n-1}h \right) \right| / |h| \\ &\leq \sum_{n=1}^\infty |a_n| \left| (z-a+h)^n - (z-a)^n - n(z-a)^{n-1}h \right| / |h| \end{aligned}$$

Now we can bound this using

$$\begin{aligned}
|(w+h)^n - w^n - nw^{n-1}h| &= \left| \left(\sum_{j=0}^{n} \binom{n}{j} w^{n-j} h^j \right) - w^n - nw^{n-1}h \right| \\
&= \left| \sum_{j=2}^{n} \binom{n}{j} w^{n-j} h^j \right| \\
&\leq |h|^2 \sum_{j=2}^{n} \binom{n}{j} |w|^{n-j} |h|^{j-2} \\
&= |h|^2 \sum_{j=2}^{n} \frac{n(n-1)}{j(j-1)} \binom{n-2}{j-2} |w|^{n-j} |h|^{j-2} \\
&\leq |h|^2 \sum_{j=2}^{n} n(n-1) \binom{n-2}{j-2} |w|^{n-j} |h|^{j-2} \\
&\leq n(n-1)|h|^2 \sum_{k=0}^{n-2} \binom{n-2}{k} |w|^{n-2-k} |h|^k \\
&= n(n-1)|h|^2 (|w| + |h|)^{n-2}
\end{aligned}$$

Hence

$$\left| \frac{f(z+h) - f(z) - g(z)h}{h} \right| \leq |h| \sum_{n=2}^{\infty} n(n-1)|a_n|(|z-a| + |h|)^{n-2}$$

and if we restrict to h so small that $|h| < (R - |z-a|)/2$, then $|z-a| + |h| < (R + |z-a|)/2$ and

$$\left| \frac{f(z+h) - f(z) - g(z)h}{h} \right| \leq |h| \sum_{n=2}^{\infty} n(n-1)|a_n| \left(\frac{R + |z-a|}{2} \right)^{n-2}$$

The latter series converges by the same reasoning used to show that $g(z)$ converges. [In fact $\sum_{n=2}^{\infty} n(n-1)a_n w^{n-2}$ has radius of convergence R and so converges absolutely for any w with $|w| < R$.] Thus

$$\lim_{h \to 0} \frac{f(z+h) - f(z) - g(z)h}{h} = 0$$

and so we have shown $f'(z) = g(z)$.

Definition 1.9 A *Laurent series* centered at $a \in \mathbb{C}$ is a series

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

By definition this series converges if both $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ and $\sum_{n=-\infty}^{-1} a_n (z-z_0)^n$ converge and the sum of the doubly infinite series is then defined as the sum of the two singly infinite sums.

For any Laurent series, there are two "numbers" R_1 and R_2

1. $0 \le R_1, R_2 \le \infty$
2. The power series converges (absolutely) for all z with $R_1 < |z-a| < R_2$
3. The power series diverges for all z with $|z-a| < R_1$ and for all z with $|z-a| > R_2$.

We can take R_2 to be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and R_1 to be the reciprocal of the radius of convergence of the power series $\sum_{n=1}^{\infty} a_{-n}w^n$.

As long as the Laurent series converges for some z, the numbers R_1 and R_2 are uniquely determined by the conditions 1. – 3. above.

Definition 1.10 By a *path* in $\mathbb{C}$ we mean a continuous function $\gamma\colon [a,b] \to \mathbb{C}$, where $a < b$ are real.

We could also consider the path as the image set $\gamma([a,b])$. Sometimes we may be careless about the distinction, but the actual parametrisation $\gamma(t)$ is significant. Very often we can allow reparametrisations $\gamma \circ \sigma\colon [\alpha,\beta] \to \mathbb{C}$, where $\sigma\colon [\alpha,\beta] \to [a,b]$ is a monotone increasing continuous bijection, as equivalent to γ.

For integration, we will restrict ourselves to *piecewise* C^1 curves γ. By a C^1 curve we mean a curve γ where $\gamma'(t) = \lim_{\mathbb{R}\ni h\to 0}(\gamma(t+h)-\gamma(t))/h$ exists (in $\mathbb{C}$) at all points $t \in [a,b]$ and $\gamma'\colon [a,b] \to \mathbb{C}$ is continuous. (At the end points $t = a$ and $t = b$ we define γ' as a one-sided limit.) To say that $\gamma\colon [a,b] \to \mathbb{C}$ is a piecewise C^1 curve means that there is a partition $a = a_0 < a_1 < a_2 < \cdots < a_n = b$ of $[a,b]$ so that the restriction of γ to each $[a_{j-1}, a_j]$ is C^1 for $j = 1, \ldots, n$. Note that such a γ must be continuous on $[a,b]$.

We then also restrict to reparametrisations by piecewise C^1 continuous bijections σ.

Examples where piecewise C^1 are easier to use than just C^1 would be where our curve is the boundary of a square or a triangle or any polygon. We can parameterize the line segment from z_0 to z_1 by

$$\gamma\colon [0,1] \to \mathbb{C} : t \mapsto (1-t)z_0 + z_1$$

and this is clearly C^1. If we string two such segments z_0 to z_1 and z_1 to z_2 together we can get a piecewise C^1 curve. To make a C^1 version we would have to reparametrise to make $\gamma' = 0$ at the corners. This is possible but not convenient.

Finally if $G \subseteq \mathbb{C}$, when we speak of a path in G we mean a continuous $\gamma\colon [a,b] \to G$.

Definition 1.11 If $\gamma\colon [a,b] \to \mathbb{C}$ is a piecewise C^1 curve and $f\colon T \to \mathbb{C}$ is continuous on a set T that contains $\gamma([a,b])$ then we define the *contour integral* of f along γ as

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$$

Strictly speaking we should do this for C^1 paths γ first and then define $\int_\gamma f(z)\,dz$ as the sum of the integrals along the finite number of C^1 restrictions of γ (to $[a_{j-1}, a_j]$ using the earlier notation) in the piecewise C^1 case. The problem is that $\gamma'(t)$ may not be defined at the transition points a_j. We know γ must have a left hand derivative and a right hand derivative at each of these points (the a_j) but the two may be different. As $\gamma'(t)$ is bounded and continuous except for these finitely many transition points, the integral makes sense using any arbitrary value in place of $\gamma'(t)$ at these transition points.

Recall 1.12 Some simple useful properties of contour integrals include:

1. If $f\colon G \to \mathbb{C}$ is continuous on an open set G and $F\colon G \to \mathbb{C}$ is an antiderivative for f on G (that is $F'(z) = f(z) \forall z \in G$) then

$$\int_\gamma f(z)\,dz = \int_a^b F'(\gamma(t))\gamma'(t)\,dt = F(\gamma(b)) - F(\gamma(b))$$

   for any piecewise C^1 path $\gamma\colon [a,b] \to G$.

   In particular, the integral depends only on the endpoints of the path. If the path is *closed* (that is if $\gamma(a) = \gamma(b)$) and f has an antiderivative, then the integral is 0.

2. $$\left|\int_\gamma f(z)\,dz\right| \leq \text{length } (\gamma) \sup_{z\in\gamma([a,b])} |f(z)|$$

   if we define the *length* of γ to be

$$\text{length } (\gamma) = \int_a^b |\gamma'(t)|\,dt$$

3. $\int_\gamma f(z)\,dz$ is unchanged by (piecewise) C^1 reparametrisations of γ.

4. Direction reversing 'reparametrisations' of γ such as $t \mapsto \tilde{\gamma}(t) = \gamma(-t)\colon [-b,-a] \to \mathbb{C}$ change the sign of the integral:

$$\int_{\tilde{\gamma}} f(z)\,dz = -\int_\gamma f(z)\,dz$$

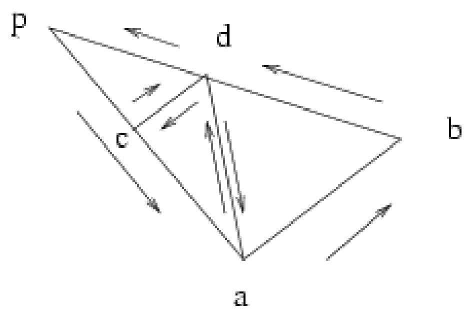
Theorem 1.13 (Cauchy's theorem for a triangle, or Goursat's theorem) *Suppose $f\colon G \to \mathbb{C}$ is analytic on an open set $G \subset \mathbb{C}$ and γ is a curve in G traversing the perimeter of a triangle exactly once. If the interior of the triangle is also in G, then*

$$\int_\gamma f(z)\,dz = 0$$

Proof. Uses the idea of repeatedly subdividing the triangle into 4 similar triangles half the size. At each step pick one where the integral is at least a quarter of the integral around the larger triangle. See Conway for the details.

Corollary 1.14 *Suppose $f\colon G \to \mathbb{C}$ is continuous on an open set $G \subset \mathbb{C}$ and analytic on $G - \{p\}$ for some one point $p \in G$ and γ is a curve in G traversing the perimeter of a triangle exactly once. If the interior of the triangle is also in G, then*

$$\int_\gamma f(z)\,dz = 0$$



Proof. The first step is to deal with the case where p is a vertex of γ. Say γ is the triangle $\triangle pab$. Assume the triangle is traversed in the direction $p \to a \to b$ and pick points c on the side pa and d on the side pb both near p.

Then

$$\begin{aligned} \int_\gamma f(z)\,dz &= \int_{\triangle pcd} f(z)\,dz + \int_{\triangle dca} f(z)\,dz + \int_{\triangle dab} f(z)\,dz \\ &= \int_{\triangle pcd} f(z)\,dz + 0 + 0 \end{aligned}$$

Thus

$$\begin{aligned} \left| \int_\gamma f(z)\,dz \right| &\leq \text{length}\,(\triangle pcd) \sup_{z \in \triangle pcd} |f(z)| \\ &\leq \text{length}\,(\triangle pcd) \sup \{|f(z)| : z \text{ inside or on } \gamma\} \end{aligned}$$

The supremum on the right is finite and thus the right hand side can be made arbitrarily small by choosing the points c and d close to p. Thus the result follows in this case.

The case when p is inside (or on) γ can be reduced to the first case by dividing the integral into the sum of 3 (or 2) integrals around triangles with p as a vertex.

When p is outside γ, there is nothing to do (by Theorem 1.13).

Definition 1.15 A set $G \subset \mathbb{C}$ is called *convex* if $z, w \in G$ and $0 \leq t \leq 1$ implies $tz + (1-t)w \in G$ (that is, if the line segment joining z and w is in G for all pairs of points $z, w \in G$).

Theorem 1.16 (Cauchy's theorem for a convex set) *Let G be an open convex set in $\mathbb{C}$. Suppose $f \colon G \to \mathbb{C}$ is continuous on G and analytic on $G - \{p\}$ for some $p \in G$. Let γ be any (piecewise C^1) closed curve in G. Then*

$$\int_\gamma f(z)\,dz = 0$$

Proof. Recall that a *closed* curve is a curve $\gamma\colon [a,b] \to \mathbb{C}$with $\gamma(a) = \gamma(b)$. Thus by earlier remarks in 1.12 it is sufficient to show that there is an antiderivative $F\colon G \to \mathbb{C}$ for f, that is an analytic function F with $F'(z) = f(z)$ for $z \in G$. Fix a point $a \in G$ and define F by $F(z) = \int_a^z f(\zeta)\,d\zeta$ where the path of integration is the straight line from a to z. To show that F is an antiderivative for f, fix $z \in G$ and consider the difference quotient

$$\begin{aligned} \frac{F(z+h)-F(z)}{h} - f(z) &= \frac{\int_a^{z+h} f(\zeta)\,d\zeta - \int_a^z f(\zeta)\,d\zeta}{h} - f(z) \\ &= \frac{\int_z^{z+h} f(\zeta)\,d\zeta}{h} - f(z) \\ &\quad \text{(using Corollary 1.14)} \\ &= \frac{\int_z^{z+h} f(\zeta) - f(z)\,d\zeta}{h} \end{aligned}$$

where the last step relies on the fact that the integral of a constant$\int_z^{z+h} f(z)\,d\zeta = f(z)h$. Now, if given any $\varepsilon > 0$, we can use continuity of f at z to find $\delta > 0$ so that $|f(\zeta) - f(z)| \le \varepsilon$ holds for all ζ with $|\zeta - z| < \delta$. But then, estimating from the above identity with the triangle inequality, we see that if $0 < |h| < \delta$, then

$$\begin{aligned} \left|\frac{F(z+h)-F(z)}{h} - f(z)\right| &\le \frac{|h|\sup_{0\le t\le 1}|f(z+th)-f(z)|}{|h|} \\ &\le \varepsilon \end{aligned}$$

This establishes that $F'(z) = f(z)$.

Definition 1.17 For γ a closed (piecewise) C^1 curve in $\mathbb{C}$ and $z \in \mathbb{C} \setminus \gamma$, we define the *index* of γ about z as

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z}\,d\zeta$$

The index is also known as the *winding number* of γ about z.

Example 1.18 Let $\gamma\colon [0,1] \to \mathbb{C}$ be the (closed) curve

$$\gamma(t) = a + re^{2\pi int}$$

(where $a \in \mathbb{C}$, $n \in \mathbb{Z}$ and $r > 0$). Informally, we can see that this curve travels n times around the circle $|z - a| = r$. Formally, we can compute

$$\begin{aligned} \mathrm{Ind}_\gamma(a) &= \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz \\ &= \frac{1}{2\pi i}\int_0^1 \frac{2\pi inre^{2\pi int}}{re^{2\pi int}}\,dt \\ &= \int_0^1 n\,dt = n \end{aligned}$$

Remark 1.19 Clearly $f(\zeta) = \frac{1}{\zeta - a}$ is analytic in $\mathbb{C} \setminus \{a\}$. If it had an antiderivative in $\mathbb{C} \setminus \{a\}$ (that is if we could find an analytic function $F(\zeta)$ with $F'(\zeta) = f(\zeta)$ there), then $\mathrm{Ind}_\gamma(a) = 0$ would hold for every closed curve γ (that did not go through a) according to the first remark we made in 1.12. From the previous example 1.18 we can then see that the index is not always zero and so that we cannot be able to find such an antiderivative $F(\zeta)$.

We will see a little later that it is almost true that $f(\zeta)$ has an antiderivative. We can define an antiderivative but we cannot do it over the whole of $\mathbb{C} \setminus \{a\}$. If we take

$$G = \mathbb{C} \setminus \{a - t : t \in \mathbb{R}, t \geq 0\}$$

we can define $F(\zeta) = \log(\zeta - a)$ for $\zeta \in G$ by taking

$$\log(\zeta - a) = \log|\zeta - a| + i\arg(\zeta - a)$$

Here the real part is the ordinary natural logarithm from real analysis (sometimes denoted by $\ln$) and the imaginary part is the argument of $\zeta - a$ and is chosen to lie in the range $(-\pi, \pi)$.

We will return to this question of defining the logarithm and why $F'(\zeta) = \frac{1}{\zeta - a}$ holds for $\zeta \in G$. Assuming this for a moment, we can see that the index $\mathrm{Ind}_\gamma(a) = 0$ for any closed curve γ in G. We can also see that if $\gamma \colon [\alpha, \beta] \to G$ begins just below the ray excluded from G at $\gamma(\alpha) = a - t - i\varepsilon$ and ends just above the ray at $\gamma(\beta) = a - t + i\varepsilon$ (where $t > 0$ and $\varepsilon > 0$ is small), then $\int_\gamma \frac{1}{\zeta - a}\, d\zeta = \log(\gamma(\beta) - a) - \log(\gamma(\alpha) - a)$ is close to being $i\pi - (-i\pi) = 2\pi i$. This allows one to see that it seems likely that if we extend γ to make it closed we get index equal to 1.

One could extend this reasoning to give a plausible explanation why a closed curve in $\mathbb{C} \setminus \{a\}$ that crosses the ray $\{a - t : t \in \mathbb{R}, t > 0\}$ several times must have index equal to an integer (the number of times the curve crosses the ray from top to bottom minus the number it crosses the other way). But is is not so easy to make a proof this way. Instead we adopt a less direct proof.

Theorem 1.20 *For any closed (piecewise) C^1 curve γ in $\mathbb{C}$ and any $z \in \mathbb{C} \setminus \gamma$, Ind$_\gamma(z)$ is an integer.*

Moreover, the function Ind$_\gamma(z)$ is constant on each connected component of $\mathbb{C} \setminus \gamma$ and identically zero in the unbounded component of $\mathbb{C} \setminus \gamma$.

Proof. Say $\gamma \colon [a, b] \to \mathbb{C}$. Define a function

$$\begin{aligned} \varphi \colon [a, b] &\to \mathbb{C} \text{ by} \\ \varphi(t) &= \exp\left(\int_a^t \frac{\gamma'(s)}{\gamma(s) - z}\, ds \right) \end{aligned}$$

From the Fundamental theorem of calculus, we can check that

$$\begin{aligned} \varphi'(t) &= \exp\left(\int_a^t \frac{\gamma'(s)}{\gamma(s) - z}\, ds \right) \frac{\gamma'(t)}{\gamma(t) - z} \\ &= \varphi(t) \frac{\gamma'(t)}{\gamma(t) - z} \end{aligned}$$

(If γ is only piecewise C^1, there will be a finite number of points where this is not true.) Then we can calculate that (again except at a finite number of points)

$$\frac{d}{dt}\left(\frac{\varphi(t)}{\gamma(t)-z}\right) = \frac{\varphi'(t)(\gamma(t)-z) - \varphi(t)\gamma'(t)}{(\gamma(t)-z)^2} = 0.$$

At the exceptional points (if any) right and left hand derivatives exist and are both 0. Thus, we can conclude that $\frac{\varphi(t)}{\gamma(t)-z} = c =$ constant. Thus $\varphi(a) = c(\gamma(a)-z)$ and $\varphi(b) = c(\gamma(b)-z)$. Since we are dealing with a closed curve, $\gamma(a) = \gamma(b)$ and so $\varphi(a) = \varphi(b)$. But $\varphi(a) = \exp(0) = 1$ (from its definition) and so we conclude that $\varphi(b) = 1$. But

$$\begin{aligned}
\varphi(b) &= \exp\left(\int_a^b \frac{\gamma'(s)}{\gamma(s)-z}\,ds\right) \\
&= \exp\left(\int_\gamma \frac{1}{\zeta - z}\,d\zeta\right) \\
&= \exp(2\pi i \mathrm{Ind}_\gamma(z))
\end{aligned}$$

Thus $\exp(2\pi i \mathrm{Ind}_\gamma(z)) = 1$ and so it follows that $\mathrm{Ind}_\gamma(z) \in \mathbb{Z}$.

To complete the proof of the result we use the fact that $\mathrm{Ind}_\gamma(z)$ is a continuous function on $\mathbb{C} \setminus \gamma$ (see Exercises). From that it follows that the index is constant on connected components (a proof is that $\{z \in \mathbb{C} \setminus \gamma : n - \frac{1}{2} < \mathrm{Ind}_\gamma(z) < n + \frac{1}{2}\}$ is both open and closed relative to $\mathbb{C} \setminus \gamma$ and so contains all of any connected component it intersects).

Finally, since $\gamma\colon [a,b] \to \mathbb{C}$ is continuous, $\gamma = \gamma([a,b])$ is a compact subset of $\mathbb{C}$. Thus there exists $R > 0$ so that $\gamma \subset D(0,R)$. If $|z| > R$, then $f(\zeta) = 1/(\zeta - z)$ is analytic on the convex set $D(0,R)$ and so Cauchy's theorem for a convex set tells us that $\mathrm{Ind}_\gamma(z) = 0$.

Example 1.21 Returning to the circle example (1.18) where we had the curve $\gamma\colon [0,1] \to \mathbb{C}$ with $\gamma(t) = a + r\exp(2n\pi i t)$, we can use the previous result to conclude that $\mathrm{Ind}_\gamma(z) = n$ for $|z-a| < r$ and $\mathrm{Ind}_\gamma(z) = 0$ for $|z| > 1$.

Theorem 1.22 (Cauchy's integral formula for a convex set) *Let $G \subset \mathbb{C}$ be an open convex set and let $f\colon G \to \mathbb{C}$ be analytic. Let γ be a closed (piecewise C^1) curve in G. Then, for $z \in G \setminus \gamma$,*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = f(z)\mathit{Ind}_\gamma(z)$$

Proof. For $z \in G \setminus \gamma$ define $g\colon G \to \mathbb{C}$ by

$$g(\zeta) = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z} \text{ if } \zeta \neq z \\ f'(z) \text{ if } \zeta = z \end{cases}$$

Then g is continuous on G and analytic on $G \setminus \{z\}$. Thus by Cauchy's theorem for a convex set (1.16), we have $\int_\gamma g(\zeta)\,d\zeta = 0$. Rearranging this we get

$$\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = \int_\gamma \frac{f(z)}{\zeta - z}\,d\zeta = f(z)\int_\gamma \frac{1}{\zeta - z}\,d\zeta = 2\pi i \mathrm{Ind}_\gamma(z) f(z).$$

Theorem 1.23 *Suppose that $f(z)$ is analytic in a disc $D(a,R)$. Let*

$$a_n = \frac{1}{2\pi i}\int_{|z-a|=r} \frac{f(z)}{(z-a)^{n+1}}\,dz$$

where r has any fixed value in the range $0 < r < R$ (for $n = 0,1,2,\ldots$) and the circle is traversed once anticlockwise. [So to be more precise we can specify the parametrisation of the path of integration as $\gamma(t) = a + re^{2\pi i t}$, $0 \le t \le 1$.] Then

$$f(z) = \sum_{n=0}^\infty a_n(z-a)^n \text{ for all } z \text{ with } |z-a| < R.$$

Moreover the values of the a_n are independent of the choice of r and we can make a stronger statement of uniqueness for the a_n: If there are coefficients b_n and any $\delta > 0$ so that

$$f(z) = \sum_{n=0}^\infty b_n(z-a)^n \text{ for } |z-a| < \delta$$

then $b_n = a_n$ for all n.

Proof. We use Theorem 1.22 together with the fact that the geometric series

$$\frac{1}{1-w} = \sum_{n=0}^\infty w^n$$

converges uniformly for $|w| \le \rho$, for any fixed $\rho < 1$. This allows us to write a term that occurs in the Cauchy integral formula as a series. Take z with $|z-a| < r$ and consider ζ with $|\zeta - a| = r$. Then

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta-a)-(z-a)} = \frac{1}{\zeta-a}\frac{1}{1-\frac{z-a}{\zeta-a}} = \frac{1}{\zeta-a}\frac{1}{1-w}$$

where $w = \frac{z-a}{\zeta-a}$ has $|w| < \frac{|z-a|}{r} = \rho < 1$. Hence

$$\frac{1}{\zeta-z} = \frac{1}{\zeta-a}\sum_{n=0}^\infty w^n = \frac{1}{\zeta-a}\sum_{n=0}^\infty\left(\frac{z-a}{\zeta-a}\right)^n = \sum_{n=0}^\infty \frac{(z-a)^n}{(\zeta-a)^{n+1}}$$

Now Theorem 1.22 tells us that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)}{\zeta-z}\,d\zeta \qquad (|z-a|<r) \\ &= \frac{1}{2\pi i}\int_{|\zeta-a|=r} f(\zeta)\sum_{n=0}^\infty \frac{(z-a)^n}{(\zeta-a)^{n+1}}\,d\zeta \\ &= \sum_{n=0}^\infty \frac{1}{2\pi i}\left(\int_{|\zeta-a|=r}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta\right)(z-a)^n \end{aligned}$$

where we make use of the uniform convergence of the series to justify exchanging the sum of the series and the integral.

Now we almost have established the first part of the statement. We now have $f(z) = \sum_{n=0}^\infty a_n(z-a)^n$ for $|z-a| < r$, but the a_n may depend on r. We can include any pre-assigned z with $|z-a| < R$ by taking r in the range $|z-a| < r < R$, but the a_n may change when we are forced to go to a bigger r. Maybe we could call them a_n^r.

We could show using Cauchy's theorem that a_n^r is independent of r, but if we prove the final uniqueness statement we will also show independence of r. So suppose we have b_n with $f(z) = \sum_{n=0}^\infty b_n(z-a)^n$ for $|z-a| < \delta$. Then we can see that $f(a) = b_0$ (giving us only one choice for b_0). Differentiating the power series (using Theorem 1.8) we get

$$f'(z) = \sum_{n=1}^\infty nb_n(z-a)^{n-1} \qquad (|z-a| < \delta$$

Putting $z = a$ gives $f'(a) = 1b_1 = b_1$ (giving only one choice for b_1). By induction we can show that

$$f^{(m)}(z) = \sum_{n=m}^\infty n(n-1)\cdots(n-m+1)b_n(z-a)^{n-1} \qquad (|z-a| < \delta$$

and $f^{(m)}(a) = m!b_m$ for all m. Thus there is only one possible choice for b_m. Applying this to $b_n = a_n^r$ we find

$$a_n^r = \frac{f^{(n)}(a)}{n!}$$

and so is independent on r.

Corollary 1.24 *If $G \subseteq \mathbb{C}$ is open and $f\colon G \to \mathbb{C}$ is analytic, then $f'\colon G \to \mathbb{C}$ is also analytic. So also are f'', $f^{(3)}$ and all higher derivatives of f defined and analytic on G.*

Proof. The second part follows immediately by induction on n once we establish that f' is automatically analytic.

We know f' is defined (because we are assuming that f is complex differentiable at every point of G) and to show it is analytic is a *local problem*. By that we mean that to show that f' is differentiable at any specific point $z \in G$ (that means showing that the second derivative makes sense) we only need to concern ourselves with the point z and the other points nearby z. [We don't have to go far away from z. Think of the limit definition of the derivative.] Since $z \in G$ and G is open we know there is some disc $D(z,r)$ of positive radius about z contained in G. We will work inside this disc, forgetting for the moment about any other part of G. [Essentially, this is a theorem which is true once we can can show it for the case $G =$ a disc.] But we know from Theorem 1.23 that in the disc the analytic function can be represented by a power series.

$$f(w) = \sum_{n=0}^\infty a_n(w-z)^n \text{ for } |w-z| < r.$$

And so we know from Theorem 1.8 that

$$f'(w) = \sum_{n=1}^{\infty} n a_n (w - z)^{n-1}$$

is analytic in that same disc. Hence $f''(z)$ exists.

Corollary 1.25 (Cauchy's formula) *If $f\colon G \to \mathbb{C}$ is analytic on an open set $G \subseteq \mathbb{C}$ and (the closed disc) $\overline{D}(a, r) \subset G$ (any $r > 0$), then*

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{|z-a|=r} \frac{f(z)}{(z-a)^{n+1}}\, dz$$

Proof. We just proved this in the course of proving Theorem 1.23.

Theorem 1.26 (Liouville's Theorem) *If $f\colon \mathbb{C} \to \mathbb{C}$ is analytic and bounded (that is, $\exists M \geq 0$ such that $|f(z)| \leq M \forall z \in \mathbb{C}$) then f is constant.*

Aside. Functions f analytic on all of $\mathbb{C}$ are traditionally call *entire functions*. Liouville's Theorem is usually stated: *bounded entire functions are constant*. It is an example of the rigidity of analytic functions.

Proof. From Theorem 1.23, f has a power series representation about the origin valid in all of $\mathbb{C}$ (the disc of infinite radius about 0),

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (z \in \mathbb{C})$$

with

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}}\, dz \quad (0 < r < \infty).$$

Estimating a_n using the triangle inequality for integrals (1.12.2), we get

$$|a_n| = \frac{1}{2\pi} \left| \int_{|z|=r} \frac{f(z)}{z^{n+1}}\, dz \right| \leq \frac{1}{2\pi}(2\pi r)\frac{M}{r^{n+1}} = \frac{M}{r^n}.$$

As this is valid for any $r > 0$ we must have $a_n = 0 \forall n > 0$ and so $f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 =$ constant.

Theorem 1.27 (Morera's theorem — a converse to Cauchy's) *If $f\colon G \to \mathbb{C}$ is continuous on an open set $G \subset \mathbb{C}$ and if $\int_\gamma f(z)\, dz = 0$ for all triangles γ contained with their interiors inside G, then f is analytic on G.*

Proof. Fixing $a \in G$, we can find a disc $D(a, R) \subset G$ ($R > 0$) (since G open) and the hypotheses imply that for any triangle γ in $R(a, R)$ [the interior of γ must be in $D(a, R)$ by convexity, and hence in G since $D(a, R) \subset G$] the restriction of f to the disc satisfies the hypotheses of the theorem. So if we prove the result for the case $G = D(a, R)$ a disc we get $f'(a)$ exists (and returning to the case of arbitrary G we have existence of the derivative $f'(a)$ at any $a \in G$).

We copy the proof of Cauchy's theorem for a convex set (1.16) and define $F\colon D(a, R) \to \mathbb{C}$ by $F(z) = \int_a^z f(\zeta)\, d\zeta$. Then we can show as before (replacing the use of Corollary 1.14 by an appeal to the hypothesis we have now) that $F'(z) = f(z) \forall z \in D(a, R)$. By Corollary 1.24, $F' = f$ is analytic on $D(a, R)$.

Definition 1.28 A *chain* Γ in $\mathbb{C}$ means a finite collection $\gamma_1, \gamma_2, \ldots, \gamma_n$ of closed curves in $\mathbb{C}$.

We will often assume that the chain is piecewise C^1, meaning that each of the curves γ_j is piecewise C^1.

We sometimes write $\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ as a formal sum, but we do not mean that any operations should be performed. There may be repetitions (the same curve can occur more than once) and the order of the curves $\gamma_1, \gamma_2, \ldots, \gamma_n$ will never be significant. The plus sign is however suggestive of the way we define integrals and lengths for chains.

Suppose $f(z)$ is continuous on Γ (we mean now Γ regarded as a set of points $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ in the plane, namely the union of the sets γ_j and these are in turn the ranges of the parametric curves γ_j). Then we define the integral of f along the chain Γ as

$$\int_\gamma f(z)\, dz = \sum_{j=1}^n \int_{\gamma_j} f(z)\, dz.$$

For $z \in \mathbb{C} \setminus \Gamma$ we define

$$\mathrm{Ind}_\Gamma(z) = \sum_{j=1}^n \mathrm{Ind}_{\gamma_j}(z)$$

and we define the length of Γ as

$$\mathrm{length}(\Gamma) = \sum_{j=1}^n \mathrm{length}(\gamma_j).$$

We often allow ourselves to use $-\gamma$ to mean the 'same' curve as γ with a parametrisation reversed in direction. Since $\int_{-\gamma} f(z)\, dz = -\int_\gamma f(z)\, dz$, and we mostly use chains for integrals we often allow cancellation of $\gamma + (-\gamma)$. [In the case of lengths, however, $\gamma + (-\gamma)$ has twice the length of γ.]

Notice that we are assuming Γ is piecewise C^1 for all these integrations.

Remarks 1.29

1. It is quite easy to check that

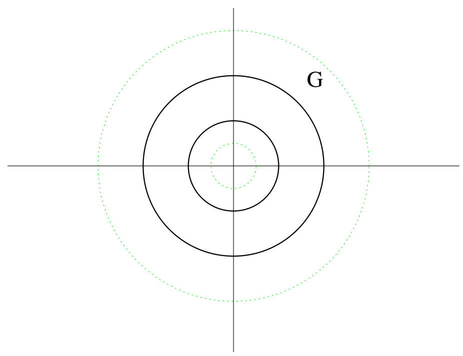
$$\left| \int_\Gamma f(z)\, dz \right| \leq \mathrm{length}(\Gamma) \sup_{z \in \Gamma} |f(z)|$$

and that various other simple properties of integrals $\int_\gamma f(z)\, dz$ along single curves also hold for integrals over chains Γ.

2. Similarly it is easy to see that $\mathrm{Ind}_\Gamma(z)$ is integer valued on $\mathbb{C} \setminus \Gamma$, constant on connected components of $\mathbb{C} \setminus \Gamma$ and zero on the unbounded component.

3. Our main use of chains Γ will be for the situation where $\Gamma \subset G$ with $G \subset \mathbb{C}$ open and where $\mathrm{Ind}_\Gamma(w) = 0$ for all $w \in \mathbb{C} \setminus G$ outside G.

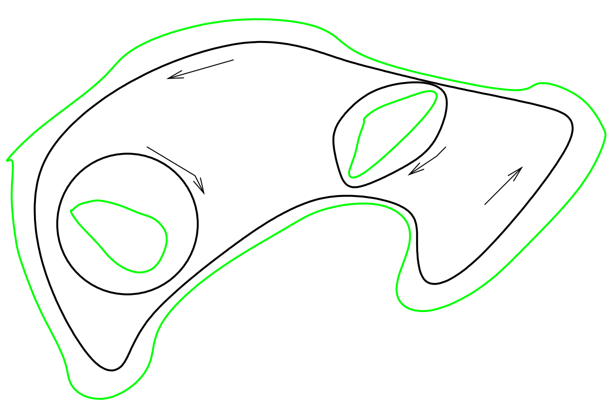
   One example might be $G = \{z \in \mathbb{C} : \frac{1}{2} < |z| < 3\}$ and $\Gamma = (-\gamma_1) + \gamma_2$ where γ_r stands for the circle of radius r about the origin traversed once anticlockwise ($\gamma_r \colon [0,1] \to \mathbb{C}$, $\gamma_r(t) = r\exp(2\pi i t)$).

   Using what we know of the index of individual circles (from 1.21) we can easily see that

   $$\mathrm{Ind}_\Gamma(z) = \begin{cases} 0 & |z| < 1 \\ 1 & 1 < |z| < 2 \\ 0 & |z| > 2 \end{cases}$$

   More complex examples could have a less regular shape but be more or less the same (one hole in G, $\Gamma = (-\sigma_1) + \sigma_2$ where the inner curve σ_1 wraps once anticlockwise tightly around the hole and the outer σ_2 takes a wider path around the hole) or one can have examples with several holes in G, maybe one outer curve and several smaller ones going around the holes. Here is a drawing of a case with 2 holes and Γ made up of 3 closed curves.

   We will be able to justify this type of example a little later.

Theorem 1.30 (Cauchy's formula — winding number version) *If $f\colon G \to \mathbb{C}$ is analytic on an open set $G \subset \mathbb{C}$ and if Γ is a (piecewise C^1) chain in G with the property that $\mathrm{Ind}_\Gamma(w) = 0 \forall w \in \mathbb{C} \setminus G$, then*

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = f(z)\mathit{Ind}_\Gamma(z) \quad \forall z \in G \setminus \Gamma.$$

Proof. We begin by defining a new function $\phi\colon G \times G \to \mathbb{C}$ on the cartesian product $G \times G \subset \mathbb{C} \times \mathbb{C} = \mathbb{C}^2$ by

$$\phi(w, z) = \begin{cases} \dfrac{f(w) - f(z)}{w - z} & \text{if } z \neq w \\ f'(z) & \text{if } z = w \end{cases}$$

We claim first that ϕ is continuous on $G \times G$ (for this think of continuity defined analogously to functions on subsets of $\mathbb{C}$ using the usual Euclidean distance on $\mathbb{C}^2$). At points (w, z) of $G \times G$ where $z \neq w$, this continuity is quite easy. $(w, z) \mapsto z$ is continuous and so is $(w, z) \mapsto w$. Hence $(w, z) \mapsto w - z$ is continuous (difference of continuous functions), as are $(w, z) \mapsto f(z)$ (composition of continuous functions) and then $(w, z) \mapsto f(w) - f(z)$ (difference). Finally $(w, z) \mapsto (f(w) - f(z))/(w - z)$ (quotient) is continuous as long as we keep away from dividing by zero. At a point $(z_0, z_0) \in G \times G$, we can argue as follows. There is some radius $R > 0$ so that $D(z_0, R) \subset G$ (as G is open) and for $w, z \in D(z_0, R)$ we can say $f(w) - f(z) = \int_z^w f'(\zeta)\,d\zeta$ (where we integrate along the straight line segment from z to w). Given any $\varepsilon > 0$, by continuity of f' at z_0 we can find $\delta > 0$ so that $|\zeta - z_0| < \delta \Rightarrow |f'(\zeta) - f'(z_0)| < \varepsilon$. Now take any (z, w) with

$$\text{distance}\,((w, z), (z_0, z_0)) = \sqrt{|w - z_0|^2 + |z - z_0|^2} < \delta.$$

We claim $|\phi(w, z) - \phi(z_0, z_0)| < \varepsilon$. If $w = z$ this is true because $\phi(w, z) = f'(z)$ and $|z - z_0| < \delta/\sqrt{2} < \delta$. If $w \neq z$,

$$|\phi(w, z) - \phi(z_0, z_0)| = \left|\frac{1}{w - z}\int_z^w (f'(\zeta) - f'(z_0))\,d\zeta\right| < \varepsilon$$

by the triangle inequality estimate for integrals. Thus the claimed continuity of ϕ is established.

Next let $H = \{z \in \mathbb{C} \setminus \Gamma : \mathrm{Ind}_\Gamma(z) = 0\}$. H is open in $\mathbb{C}$. (Since Γ has to be compact (finite union of closed curves, each compact), $\mathbb{C} \setminus \Gamma$ is open in $\mathbb{C}$ and so are its connected components. H is a union of certain connected components of $\mathbb{C} \setminus \Gamma$ and so is also open.)

Our hypotheses imply $\mathbb{C} \setminus G \subset H$ and so $G \cup H = \mathbb{C}$. We now define $g\colon \mathbb{C} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \displaystyle\int_\Gamma \phi(w, z)\,dw & \text{for } z \in G \\ \displaystyle\int_\Gamma \frac{f(w)}{w - z}\,dw & \text{for } z \in H \end{cases}$$

We make a sequence of claims about g culminating in the result.

1. g is well-defined and analytic on $\mathbb{C}$

Verification. To show that g is unambiguously defined we need to check that the two formulae agree on $G \cap H$. For $z \in G \cap H$, we have

$$\begin{aligned}
\int_\Gamma \phi(w,z)\,dw &= \int_\Gamma \frac{f(w)-f(z)}{w-z}\,dw \\
&= \int_\Gamma \frac{f(w)}{w-z}\,dw - f(z)\int_\Gamma \frac{1}{w-z}\,dw \\
&= \int_\Gamma \frac{f(w)}{w-z}\,dw - f(z)(2\pi i)\mathrm{Ind}_\Gamma(z) \\
&= \int_\Gamma \frac{f(w)}{w-z}\,dw \text{ since } z \in H
\end{aligned}$$

To show g is entire we show separately that it is analytic on G and on H. In both cases the proof is similar. We establish that g is continuous (Exercise to show that a function defined by integrating a continuous function of two variables z, w around a curve in the w variable gives a continuous function of the parameter z.) Then we use Morera's theorem and Fubini's theorem to show that the result is analytic. For example if γ is a triangle in G with its interior also in G, then

$$\begin{aligned}
\int_\gamma g(z)\,dz &= \int_\gamma \int_\Gamma \phi(w,z)\,dw\,dz \\
&= \int_\Gamma \int_\gamma \phi(w,z)\,dz\,dw \\
&= 0
\end{aligned}$$

because for any fixed $w \in \Gamma$, $z \mapsto \phi(w,z)$ is continuous on G and analytic on $G \setminus \{w\}$ and so the integral $\int_\gamma \phi(w,z)\,dz = 0$ by the Corollary 1.14 to Cauchy's theorem for a triangle.

2. g is also bounded and hence constant by Louville's theorem

   Verification. Compactness of Γ implies it is bounded and so there is a disc $D(0,R)$ of some finite radius $R > 0$ that contains Γ ($\Gamma \subset D(0,R)$). Now $|z| > R$ implies z belongs in the unbounded component of $\mathbb{C} \setminus \Gamma$, hence where $\mathrm{Ind}_\Gamma(z) = 0$, that is $z \in H$.

   Thus for $|z| > R$ we have

$$|g(z)| = \left| \int_\Gamma \frac{f(w)}{w-z}\,dw \right| \leq \text{length}\,(\Gamma) \sup_{w\in\Gamma} |f(w)| \frac{1}{|z|-R}$$

   For $|z| > R+1$ we then have a fixed bound for $g(z)$ of $M_1 = \text{length}\,(\Gamma)\sup_{w\in\Gamma}|g(w)|$ and for $|z| \leq R+1$ we can use compactness to say $M_2 = \sup_{|z|\leq R+1}|g(z)| < \infty$. For all $z \in \mathbb{C}$ we then have

$$|g(z)| \leq \max(M_1, M_2)$$

   and g is bounded. By Louville, g must be constant.

3. The constant is 0

   Verification. By the estimate just above we have

$$|g(0)| = |g(z)| \leq \frac{M_1}{|z| - R} \quad \text{for } |z| > R$$

   and letting $|z| \to \infty$ we get $g(0) = 0$.

4. The integral formula follows

   Verification. For $z \in G \setminus \Gamma$ we have $g(z) = 0$, which means

$$\begin{aligned} 0 &= \int_\Gamma \phi(w, z)\, dw = \int_\Gamma \frac{f(w) - f(z)}{w - z}\, dw \\ &= \int_\Gamma \frac{f(w)}{w - z}\, dw - f(z) \int_\Gamma \frac{1}{w - z}\, dw \\ &= \int_\Gamma \frac{f(w)}{w - z}\, dw - f(z)(2\pi i)\mathrm{Ind}_\Gamma(z) \end{aligned}$$

Corollary 1.31 (Cauchy's theorem — winding number version) *If $f\colon G \to \mathbb{C}$ is analytic on an open set $G \subset \mathbb{C}$ and if Γ is a (piecewise C^1) chain in G with the property that Ind$_\Gamma(w) = 0 \forall w \in \mathbb{C} \setminus G$, then*

$$\frac{1}{2\pi i} \int_\Gamma f(z)\, dz = 0$$

Proof. Apply Cauchy's formula 1.30 above to $w \mapsto f(w)(w - z)$ in place of $f(w)$ (for any $z \in G \setminus \Gamma$).

Definition 1.32 A *simple closed curve* in $\mathbb{C}$ is a closed curve $\gamma\colon [a, b] \to \mathbb{C}$ such that $\gamma_{[a,b)}$ is injective. [In other words the curve has no self-intersections except that it closes — same beginning point and end point.]

We rule out the trivial case of $a = b$ (curve has only one point).

Theorem 1.33 (Jordan curve theorem) *If γ is a simple closed curve in $\mathbb{C}$, then $\mathbb{C} \setminus \gamma$ has exactly two connected components, one unbounded component which we will call the* outside *and one bounded which we will call the* inside.

Proof. (Omitted as it is rather difficult.) One book that proves it is [1].

Theorem 1.34 (Cauchy's theorem — for simple closed curves) *Let $f\colon G \to \mathbb{C}$ be an analytic function on an open set $G \subset \mathbb{C}$ and let γ be a (piecewise C^1) simple closed curve in G which has its inside also contained in G. Then*

$$\int_\gamma f(z)\, dz = 0$$

Proof. We can deduce this from the winding number version Theorem 1.31 and the fact that $\mathrm{Ind}_\gamma(z) = 0$ for z in the unbounded component of $\mathbb{C} \setminus \gamma$ (that is for z outside γ). If $z \in \mathbb{C} \setminus G$ then z is outside γ by our hypothesis.

Theorem 1.35 *Let γ be a simple closed piecewise C^1 curve in $\mathbb{C}$ and z a point of the inside of γ. Then $\mathrm{Ind}_\gamma(z)$ is either $+1$ or -1.*

Proof. Omitted. See [1].

Definition 1.36 Let γ be a simple closed piecewise C^1 curve in $\mathbb{C}$. We say that γ is *oriented anticlockwise* if $\mathrm{Ind}_\gamma(z) = 1$ for z inside γ. Otherwise ($\mathrm{Ind}_\gamma(z) = -1$ for z inside γ) we call γ *oriented clockwise*.

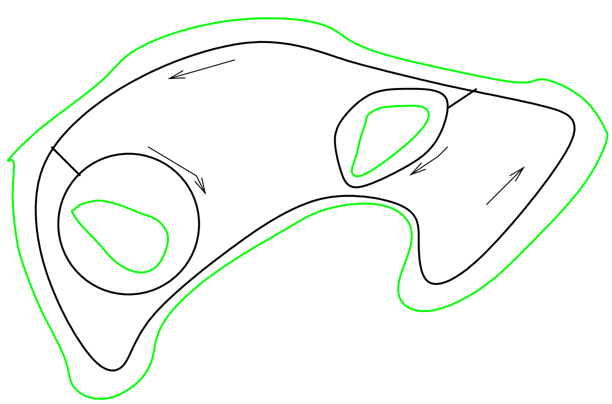
Theorem 1.37 (Cauchy's integral formula — simple closed curve version) *Let $G \subset \mathbb{C}$ be open, $f \colon G \to \mathbb{C}$ analytic and γ an anticlockwise (piecewise C^1) simple closed curve in G with its inside also contained in G. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z) \text{ for } z \text{ inside } \gamma$$

Proof. This is a consequence of the facts about simple closed curves above and the winding number version of the Theorem (1.30).

Remarks 1.38 We can use the terminology of Definition 1.36 to justify the rough picture at the end of Remarks 1.29 provided we belive that the definition corresponds to our picture of anticlockwise curves.

In cases like those examples of Remarks 1.29 it is possible to use the versions of Cauchy Theorem and Cauchy's Integral Formula for simple closed curves to justify the winding number versions (1.31 and 1.30). One constructs a simple closed curve (or maybe several such) by building narrow bridges between the 'outer' curves and the inner ones. Apply the simple closed curve theorems, let the width of the 'bridge' tend to zero and cancel out the contributions from integrating back and forth across the bridges.



We proceed now to another way to state something like 'the points between γ_0 and γ_1 are in G' (or that both γ_0 and γ_1 wind equally often around each point of the complement of G) even when there is no obvious way to describe what the points between might be. The idea of homotopy is used extensively in algebraic topology. The formal definition is meant to capture the idea that we can move one closed curve onto another without breaking the curve and without ever going outside a given set G.

We keep our curves parametrised by the same interval $[a, b]$ (we could standardise it as $[0, 1]$ but we don't).

Definition 1.39 Let $\gamma_0, \gamma_1 \colon [a, b] \to G$ be two closed curves in a set $G \subset \mathbb{C}$. Then we say that γ_0 *is homotopic to* γ_1 *in G* if there exists a continuous map $H \colon [a, b] \times [0, 1] \to G$ satisfying:

1. $H(t, 0) = \gamma_0(t) \forall t \in [a, b]$
2. $H(t, 1) = \gamma_1(t) \forall t \in [a, b]$
3. $H(a, s) = H(b, s) \forall s \in [0, 1]$

Such a map H is called a *homotopy* from γ_0 to γ_1.

Note that for fixed $s \in [0, 1]$ we have a closed curve $\gamma_s \colon [a, b] \to G$ given by $\gamma_s(t) = H(t, s)$ (closed by the third condition). The way to think of it is that these closed curves γ_s vary continuously from γ_0 to γ_1 (as s varies from 0 to 1).

Examples 1.40

1. Let $G = \mathbb{C} \setminus \{0\}$ and let $\gamma_0, \gamma_1 \colon [0, 1] \to G$ be $\gamma_0(t) = \exp(2\pi i t)$, $\gamma_1(t) = 2\exp(2\pi i t)$.

   Then a homotopy $H \colon [0, 1] \times [0, 1] \to G$ can be given as $H(t, s) = (1 + s)\exp(2\pi i t)$.

2. If $\gamma_0, \gamma_1 \colon [a, b] \to \mathbb{C}$ are any pair of closed curves in $\mathbb{C}$ parametrised by the same interval $[a, b]$, then they are homotopic in $\mathbb{C}$ via $H(t, s) = (1 - s)\gamma_0(t) + s\gamma_1(t)$.

3. If $\gamma_0, \gamma_1 \colon [a, b] \to G$ are any pair of closed curves in a convex set $G \subset \mathbb{C}$, then they are homotopic in G (by the same H as in the previous example).

   The existence of a homotopy becomes open to question when the shape of G is more complicated.

Definition 1.41 If $\gamma \colon [a, b] \to G$ is a closed curve in a set $G \subset \mathbb{C}$, we say that γ is *null homotopic* in G if γ is homotopic in G to a constant curve, such as the curve $\sigma \colon [a, b] \to G$ with $\sigma(t) \equiv \gamma(a)$.

Theorem 1.42 (Cauchy's theorem — homotopy version) *Let $f \colon G \to \mathbb{C}$ be analytic on an open set $G \subset \mathbb{C}$ and let $\gamma_0, \gamma_1 \colon [a, b] \to G$ be two (piecewise C^1) closed curves in G which are homotopic in G. Then*

$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz.$$

Proof. We will see that a complication arises in the proof because, although we assume the curves γ_0, γ_1 are C^1, we do not assume that the homotopy is a differentiable map. This means that the intermediate curves involved in the homotopy are not necessarily C^1 and so we cannot necessarily integrate along them.

Notice first that if G is convex, then we already know $\int_{\gamma_0} f(z)\,dz = 0 = \int_{\gamma_1} f(z)\,dz$ by Cauchy's theorem for a convex set. So there is nothing to do. In particular $G = \mathbb{C}$ is ok and when $G \neq \mathbb{C}$ we make use of the following observation.

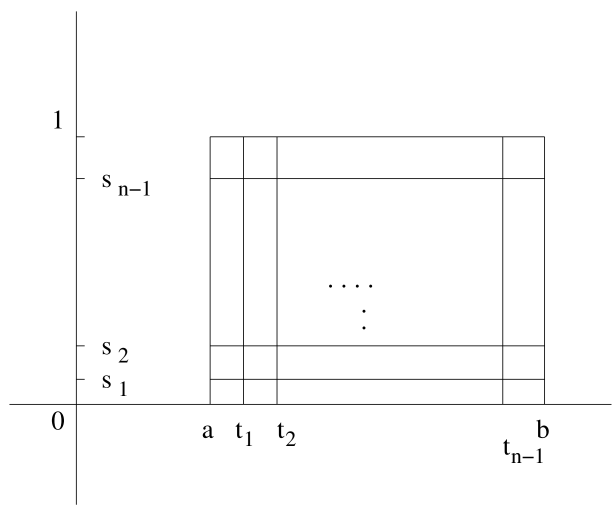
Let $H\colon [a,b] \times [0,1] \to G$ be a particular homotopy from γ_0 to γ_1. Since the range $H([a,b] \times [0,1])$ is compact, it has a positive distance to the closed complement $\mathbb{C} \setminus G$,

$$\varepsilon = \inf\{|z - w| : z \in H([a,b] \times [0,1]), w \in \mathbb{C} \setminus G\} > 0.$$

Next H is uniformly continuous on the rectangle $[a,b] \times [0,1]$ (because the rectangle is compact and H is continuous) and so we can find $\delta > 0$ so that

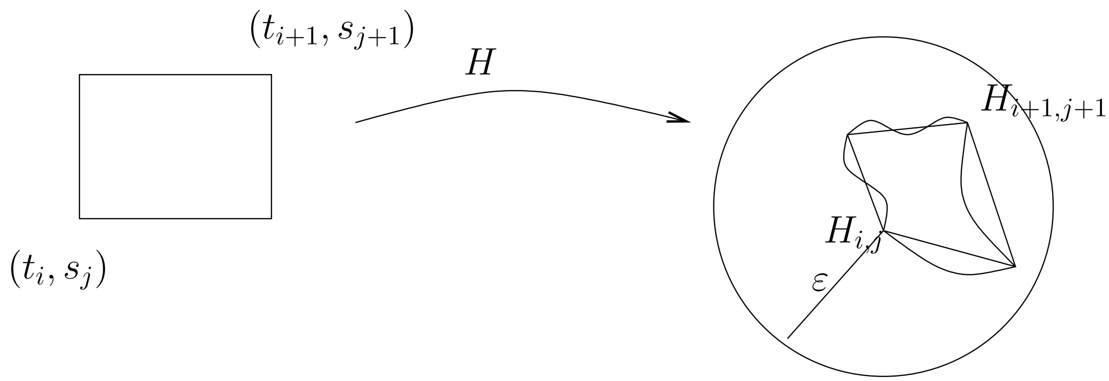
$$(t_1, s_1), (t_2, s_2) \in [a,b] \times [0,1], \text{dist}\,((t_1,s_1),(t_2,s_2)) < \delta \Rightarrow |H(t_1,s_1) - H(t_2,s_2)| < \varepsilon.$$

Now divide $[a,b] \times [0,1]$ into a grid of rectangles each of diameter $< \delta$.



To explain this more formally in symbols, choose a partition $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$ of $[a,b]$ with all $t_j - t_{j-1} < \delta/\sqrt{2}$ (for example we can have $t_j = a + j(b-a)/n$ and n so large that $(b-a)/n < \delta/\sqrt{2}$) and another partition $0 = s_0 < s_1 < s_2 < \cdots < s_{n-1} < s_n = 1$ of $[0,1]$ with all $s_j - s_{j-1} < \delta/\sqrt{2}$ (for example we could have $s_j = j/n$ as long as n is also large enough that $1/n < \delta/\sqrt{2}$).

Let (t_i, s_j) $(0 \leq i, j \leq n)$ denote the grid points in the rectangle and $H_{ij} = H(t_i, s_j)$ the corresponding image points in G. Now the fact that the small grid rectangle from with bottom left corner (t_i, s_j) (more exactly the rectangle $[t_i, t_{i+1}] \times [s_j, s_{j+1}]$ for $0 \leq i, j < n$) has diagonal of length less than δ implies that its image under H is entirely contained in the disc $D(H_{ij}, \varepsilon) \subset G$.



Consider for a moment this single small rectangle of the grid. We would like to be able to say that the integral of f around the contour which is the image of the perimeter of the rectangle under H is 0 (by Cauchy's theorem for the convex set $D(H_{i,j}, \varepsilon)$) but we are not necessarily justified in this claim because we cannot be sure that the contour is piecewise C^1.

Instead we consider the closed curve made of 4 straight line segments $H_{i,j} \to H_{i+1,j} \to H_{i+1,j+1} \to H_{i,j+1} \to H_{i,j}$. We call this contour $R_{i,j}$ ($0 \leq i, j < n$). Now, we can say

$$\int_{R_{i,j}} f(z)\, dz = 0$$

and then we can add all these up to get

$$\sum_{i,j=0}^{n-1} \int_{R_{i,j}} f(z)\, dz = 0.$$

When we express these integrals along $R_{i,j}$ as the sum of 4 integrals along line segments, we will find many line segments that are integrated along twice, one in each direction. For $0 \leq i < n$ and $0 < j < n$, the segment $H_{i,j} \to H_{i+1,j}$ occurs in $R_{i,j}$ and the segment $H_{i+1,j} \to H_{i,j}$ occurs in $R_{i,j-1}$. For $0 < i < n$ and $0 \leq j < n$, the segment $H_{i,j+1} \to H_{i,j}$ occurs in $R_{i,j}$ and the segment $H_{i,j} \to H_{i,j+1}$ occurs in $R_{i-1,j}$. The segment $H_{0,j+1} \to H_{0,j}$ occurs in $R_{0,j}$ ($0 \leq j < n$) while the segment $H_{n,j} \to H_{n,j+1}$ occurs in $R_{n-1,j}$. By the properties of a homotopy $H_{0,j} = H(t_0, s_j) = H(a, s_j) = H(b, s_j) = H(t_n, s_j) = H_{n,j}$ and so these last pair of segments are the same segment in different directions. After all the cancellations, we end up with

$$\sum_{i=0}^{n-1} \int_{[H_{i,0}, H_{i+1,0}]} f(z)\, dz - \sum_{i=0}^{n-1} \int_{[H_{i,n}, H_{i+1,n}]} f(z)\, dz = 0$$

Now, the straight line segment $[H_{i,0}, H_{i+1,0}]$ and the restriction of $\gamma_0(t) = H(t, 0)$ to $t \in [t_i, t_{i+1}]$ are both curves in $D(H_{i,0}, \varepsilon) \subset G$ with the same start and end. Hence the integrals of f along them is the same and

$$\sum_{i=0}^{n-1} \int_{[H_{i,0}, H_{i+1,0}]} f(z)\, dz = \int_{\gamma_0} f(z)\, dz.$$

Similarly

$$\sum_{i=0}^{n-1} \int_{[H_{i,n}, H_{i+1,n}]} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

We conclude

$$\int_{\gamma_0} f(z)\,dz - \int_{\gamma_1} f(z)\,dz = 0.$$

Corollary 1.43 (Cauchy's theorem) *Let $f\colon G \to \mathbb{C}$ be analytic on an open set $G \subset \mathbb{C}$ and let γ be a (piecewise C^1) closed curve in G which is null homotopic in G. Then*

$$\int_{\gamma} f(z)\,dz = 0.$$

Example 1.44 The curve $\gamma\colon [0,1] \to \mathbb{C} \setminus \{0\}$ with $\gamma(t) = \exp(2\pi i t)$ is **not** null homotopic in $\mathbb{C} \setminus \{0\}$.

Proof. We know $\int_\gamma \frac{1}{z}\,dz = 2\pi i \neq 0$. Now $f(z) = 1/z$ is analytic in $\mathbb{C} \setminus \{0\}$. By the Corollary γ cannot be null homotpic in $\mathbb{C} \setminus \{0\}$.

Corollary 1.45 (Cauchy's integral formula — homotopy version) *Let $f\colon G \to \mathbb{C}$ be analytic on an open set $G \subset \mathbb{C}$ and let γ be a (piecewise C^1) closed curve in G which is null homotopic in G. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = f(z)\mathit{Ind}_\gamma(z) \quad \forall z \in G \setminus \gamma.$$

Proof. By corollary 1.43, for $w \in G \setminus \gamma$ we have

$$\mathrm{Ind}_\gamma(w) = \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z}\,d\zeta = 0 \quad \forall w \in \mathbb{C} \setminus G.$$

The result follows by the winding number version Theorem 1.30.

References

[1] J. Dieudonné, *Foundations of Modern Analysis* Volume 1, Academic Press.

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