414 2003–04

Before we launch into complex analysis, it may be helpful to look back for a moment at the history of the subject. Calculus goes back to Leibniz and Newton, but calculus with complex-valued functions of a complex independent variable is not quite so old. There was a debate on the validity of complex numbers (and the imaginary i) but there was also a long process of discovery of the concept of a function or map as we know it today.

At one time functions were regarded as things that were given by a 'proper' formula and analytic functions fitted into this idea.

Cauchy (1769–1857) developed contour integrals and the idea of analytic functions as complex differentiable functions with a continuous derivative.

Riemann (1826–1866) took a geometrical view and looked as complex functions as mappings or transformations.

Weierstrass (1815–1897) is responsible for a formal approach, the ϵ-δ way of dealing with things and the theory of uniform convergence of power series.

Chapter 0: Basics

Notation 0.1 $\mathbb{C}$ will denote the complex numbers. For $z \in \mathbb{C}$ we will often write $z = x + iy$ with $x, y \in \mathbb{R}$ the real and imaginary parts of z and $i^2 = -1$.

The modulus (or absolute value) of such a z is $|z| = \sqrt{x^2 + y^2}$. *Properties:* $|z + w| \leq |z| + |w|$ (triangle inequality), $|zw| = |z|\,|w|$ $(z, w \in \mathbb{C})$.

The *complex conjugate* of $z = x+iy$ is $\bar{z} = x-iy$. *Properties:* $\overline{z + w} = \bar{z}+\bar{w}$, $\overline{zw} = \bar{z}\,\bar{w}$, $z\bar{z} = |z|^2$.

The usual (Euclidean) distance between points $z, w \in \mathbb{C}$ is $d(z, w) = |z - w|$. d is also called a *metric*. *Properties:* $d(z, w) \geq 0$ with equality if and only if $z = w$, $d(z, w) = d(w, z)$, $d(z, w) \leq d(z, v) + d(v, w)$ (triangle inequality).

$$D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

is the open disk about $z_0 \in \mathbb{C}$ of radius $r > 0$.

$$\bar{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

is the closed disk.

Open and closed subsets 0.2 A set $G \subseteq \mathbb{C}$ is *open* if each $z \in G$ is an *interior point* of G.

A point $z \in G$ is called an interior point of G if there is a disk $D(z, r) \subset G$ with $r > 0$.

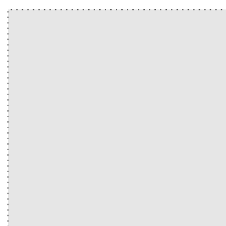
Picture for an open set: G contains **none** of its 'boundary' points.

Any union $G = \bigcup_{i \in I} G_i$ of open sets $G_i \subseteq \mathbb{C}$ is open (I any index set, arbitrarily large).

$F \subseteq \mathbb{C}$ is *closed* if its complement $\mathbb{C} \setminus F$ is open.

Picture for a closed set: F contains **all** of its 'boundary' points.

Note that open and closed are opposite extremes. There are plenty of sets which are neither open nor closed. For example $\{x + iy : 0 \leq x, y < 1\}$ is a square in the plane with some of the 'boundary' included and some not. It is neither open nor closed.



Any intersection $F = \bigcup_{i \in I} F_i$ of closed sets $F_i \subset \mathbb{C}$ is closed.

Finite intersection $G_1 \cap G_2 \cap \cdots \cap G_n$ of open sets are open.

Finite unions of closed sets are closed.

Interiors and closures 0.3 For any set $E \subseteq \mathbb{C}$, the interior E° is the set of all its interior points.

$$E^\circ = \{z \in E : \exists r > 0 \text{ with } D(z, r) \subseteq E\}$$

is the largest open subset of $\mathbb{C}$ contained in E. Also

$$E^\circ = \bigcup \{G : G \subseteq E, G \text{ open in } \mathbb{C}\}$$

Picture: E° is E minus all its 'boundary' points.

The closure of E is

$$\bar{E} = \bigcap \{F : F \subset \mathbb{C}, E \subset F \text{ and } F \text{ closed}\}$$

and it is the smallest closed subset of $\mathbb{C}$ containing E.

Picture: $\bar{E}$ is E with all its 'boundary' points added.

Properties: $\bar{E} = \mathbb{C} \setminus (\mathbb{C} \setminus E)^\circ$ and $E^\circ = \mathbb{C} \setminus (\overline{\mathbb{C} \setminus E})$.

Boundary 0.4 *The* boundary ∂E *of a set* $E \subseteq \mathbb{C}$ *is defined as* $\partial E = \bar{E} \setminus E^\circ$.

This formal definition makes the previous informal pictures into facts.

Relatively open and closed 0.5 *If* $R \subseteq T \subseteq \mathbb{C}$ *is a subset of a subset* T *of* $\mathbb{C}$*, then* R *is called* open in T *(or* open relative to T*) if for each* $z \in R$ *it is possible to find some* $r > 0$ *so that* $D(z, r) \cap T \subset R$.

Note that for each $z \in R$ we have a choice of $r = r_z > 0$ that depends on $z \in R$ so that $D(z, r_z) \cap T \subset R$. Taking $G = \bigcup_{z \in R} D(z, r_z)$ we find an open set $G \subset \mathbb{C}$ with $R = T \cap G$. This is an equivalent condition for $R \subset T$ to be open in T.

$S \subseteq T$ is called *closed in* T if $T \setminus S$ is open in T. Equivalently, if $S = T \cap F$ for $F \subset \mathbb{C}$ closed.

Picture: $S \subset T$ is closed in T contains all of its boundary points that are in T.

This can also be used a pictorial explanation of relatively open $R \subset T$: R does not contain any of the boundary of $S = T \setminus R$.

Connected 0.6 *A subset* $T \subseteq \mathbb{C}$ *is called* connected *if the only subsets* $X \subseteq T$ *that are both open in* T *and closed in* T *are* $X = \emptyset$ *and* $X = T$.

Equivalently, if it is not possible to decompose $T = T_1 \cup T_2$ with $T_1 \cap T_2 = \emptyset$, $T_1 \neq \emptyset$, $T_2 \neq \emptyset$ and both T_1 and T_2 open in T.

Proposition 0.7 *If* $T_1, T_2 \subseteq \mathbb{C}$ *are each connected and* $T_1 \cap T_2 \neq \emptyset$*, then* $T_1 \cup T_2$ *is connected.*

More generally, if $\{T_i : i \in I\}$ *is a family of connected subsets* $T_i \subseteq \mathbb{C}$ *and* $T_i \cap T_j \neq \emptyset$ *for each* $i, j \in I$*, then* $\bigcup_{i \in I} T_i$ *is connected.*

Definition 0.8 *If* $T \subseteq \mathbb{C}$ *is a set and* $z \in \mathbb{C}$*, then the* connected component *of* z *in* T *is*

$$\bigcup \{X : X \subseteq T, z \in X \text{ and } X \text{ connected}\}$$

Remark 0.9 We can define an equivalence relation on any set $T \subseteq \mathbb{C}$ by defining $z \sim w$ (for $z, w \in T$) if there is some connected $X \subseteq T$ with both $z, w \in X$.

The connected component of a point $z \in T$ is then the equivalence class of z under this equivalence relation.

Using the general theory of equivalence relations, it follows that the connected components of any $T \subseteq \mathbb{C}$ partition T. (That is, any two connected components of T are either identical or disjoint.)

Example 0.10 *1.* $T = D(0,1) \cup D(2,1)$ *is not connected.*

2. We will see that $D(0,1) \cup D(2,1) \cup \{1\}$ *is connected.*

Theorem 0.11 *Intervals in* $\mathbb{R}$ *are connected.*

Continuity 0.12 *If* $f : T \to \mathbb{C}$ *is a function on a subset* $T \subseteq \mathbb{C}$*, then* f *is called* continuous at a point $z_0 \in T$ *if for each* $\varepsilon > 0$ *it is possible to find* $\delta > 0$ *so that*

$$z \in T,\ |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$$

$f : T \to \mathbb{C}$ *is called* continuous *if it is continuous at each point* $z_0 \in T$*.*

Proposition 0.13 *If* $T \subseteq \mathbb{C}$ *and* $f : T \to \mathbb{C}$*, then* f *is continuous if and only if it satisfies the following condition: for each open set* $U \subset \mathbb{C}$*, its inverse image* $f^{-1}(U) = \{z \in T : f(z) \in U\}$ *is open in* T*.*

Proposition 0.14 *If* $T \subseteq \mathbb{C}$*, then* T *is connected if and only if there is no continuous function* $f : T \to \mathbb{R}$ *with range the two point set* $\{0, 1\}$*.*

Equivalently if the only continuous functions $f : T \to \mathbb{R}$ *with range* $f(T) \subset \{0, 1\}$ *are constant.*

Proof. Exercise.

Theorem 0.15 *If* $f : T \to \mathbb{C}$ *is continuous and* $T \subseteq \mathbb{C}$ *is connected, then* $f(T) = \{f(z) : z \in T\}$ *is connected. (Continuous images of connected sets are connected.)*

Proof. Exercise. (Not so hard using the previous result.)

Path Connectedness 0.16 By a *path* in $\mathbb{C}$, we mean a continuous function $\gamma : I \to \mathbb{C}$ from an interval $I \subseteq \mathbb{R}$.

We say that a set $T \subseteq \mathbb{C}$ is *path connected* if for each pair of points $z_0, z_1 \in T$ it is possible to find a path $\gamma : [0, 1] \to T$ in T with starting point $\gamma(0) = z_0$ and ending point $\gamma(1) = z_1$.

Proposition 0.17 *Path connected subsets* $T \subseteq \mathbb{C}$ *are connected.*

Proof. If $\gamma\colon [0,1] \to T$ is a path, then $\gamma([0,1)$ is a connected set by Theorems 0.11 and 0.15. Thus $\gamma(0)$ and $\gamma(1)$ belong in the same connected component of T for any such path.

If T is path connected, this argument shows that every pair of points of T belong in the same connected component and so there is only one connected component (or no connected components at all if T is empty). Thus T is connected.

Remark 0.18 This proposition can be used to show that discs are connected sets. For example if $z \in D(z_0, r)$ then the path $\gamma\colon [0,1] \to D(z_0, r)$ given by $\gamma(t) = (1-t)z_0 + tz$ is a path in the disc joining z_0 to z.

One can also show that $D(0,1) \cup D(2,1) \cup \{1\}$ is path connected and so connected, justifying an earlier example.

Theorem 0.19 *If $G \subset \mathbb{C}$ is open and connected, then G is path connected.*

Proof. Exercise.

Limits 0.20 We will find it convenient to have the idea of a punctured disc. A punctured (open) disc is a disc $D(z,r) \setminus \{z\}$ minus its center.

If $f\colon T \to \mathbb{C}$ is a function defined on some set $T \subseteq \mathbb{C}$ which includes some punctured disc $D(z_0, r) \setminus \{z_0\}$ of positive radius about $z_0 \in \mathbb{C}$ and if $\ell \in \mathbb{C}$, then to say that the limit of f as z approaches z_0 is ℓ, or in symbols,

$$\lim_{z \to z_0} f(z) = \ell$$

means the following:

for each $\epsilon > 0$ it is possible to find $\delta > 0$ so that

$$z \in \mathbb{C},\ 0 < |z - z_0| < \delta \Rightarrow |f(z) - \ell| < \epsilon$$

An equivalent condition is that: for each sequence $(z_n)_{n=1}^{\infty}$ in $G \setminus \{z_0\}$ with $\lim_{n\to\infty} z_n = z_0$ we have $\lim_{n\to\infty} f(z_n) = \ell$.

The catch is that we need ϵ's to define limits of sequences. To say $\lim_{n\to\infty} w_n = \ell$ means:

for each $\epsilon > 0$ it is possible to find $N \in \mathbb{N}$ so that

$$n \in \mathbb{N}, n > N \Rightarrow |w_n - \ell| < \epsilon.$$

Proposition 0.21 *If $G \subset \mathbb{C}$ is open and $f\colon G \to \mathbb{C}$, then f is continuous at a point $z_0 \in G$ if and only if $\lim_{z\to z_0} f(z) = f(z_0)$.*

Remark 0.22 One can show that the limit of a sum is the sum of the limits (provided the individual limits make sense). More symbolically,

$$\lim_{z\to z_0} f(z) + g(z) = \lim_{z\to z_0} f(z) + \lim_{z\to z_0} g(z).$$

Similarly

$$\lim_{z\to z_0} f(z)g(z) = (\lim_{z\to z_0} f(z))(\lim_{z\to z_0} g(z))$$

if both individual limits exist.

We also have the result on limits of quotients,

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)}$$

provided $\lim_{z\to z_0} g(z) \neq 0$. In short the limit of a quotient is the quotient of the limits provided the limit in the denominator is not zero.

There is also a theorem on limits of compositions, which needs continuity. If $\lim_{z\to z_0} f(z) = \ell$ and $g\colon T \to \mathbb{C}$ is defined on a set T that has ℓ as in interior point and if g is continuous at ℓ, then

$$\lim_{z\to z_0} g(f(z)) = g(\ell).$$

An important basic example of continuity is provided by polynomial functions $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. We have $\lim_{z\to z_0} p(z) = p(z_0)$ (for any polynomial p and any $z_0 \in \mathbb{C}$).

Compactness 0.23 Let $T \subseteq \mathbb{C}$ be a subset of the complex plane. An *open cover* of T is a family $\mathcal{U}$ of open subsets of $\mathbb{C}$ such that

$$T \subseteq \bigcup \{U : U \in \mathcal{U}\}$$

A subfamily $\mathcal{V} \subseteq \mathcal{U}$ is called a subcover of $\mathcal{U}$ if $\mathcal{V}$ is also a cover of T.

T is called *compact* if each open cover of T has a finite subcover.

T is called *bounded* if there exists $R \geq 0$ with $T \subseteq \bar{D}(0, R)$.

One way to state the *Heine-Borel theorem* is that a subset $T \subseteq \mathbb{C}$ is compact if and only if it is both (1) closed and (2) bounded.

Continuous images of compact sets are compact: $T \subseteq \mathbb{C}$ compact, $f\colon T \to \mathbb{C}$ continuous implies $f(T)$ compact.