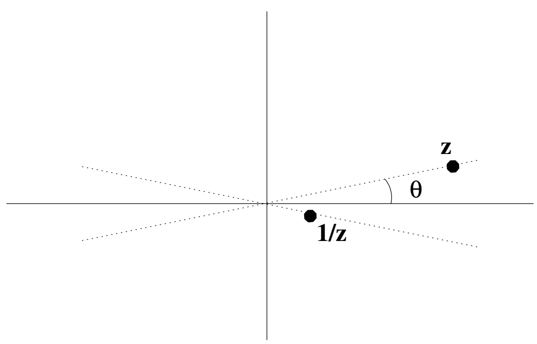
3E1 Trinity Term Tutorial sheet 9
[April 9 – 14, 2003]

Name: Solutions

1. Show that the map $f(z) = 1/z$ maps lines though the origin to (other) lines through the origin if you omit the origin itself. [Hint: what is the argument of $1/z$?]

   Solution: On a line through the origin, the argument of z is constant. To be more precise the argument of z will be constant $= \theta$ (say) along a ray (half-line) from the origin and will be $= \theta + \pi$ or $\pi - \theta$ on the opposite ray.

   Since the argument of $1/z$ is $-\arg z$ (up to possibly adding multiples of 2π to bring the argument back into the range $(-\pi, \pi]$), we can see that the argument of $1/z$ will also be constant on the image rays. On the ray where $\arg z = \theta$ we will have $\arg(1/z) = -\theta + 2n\pi$. On the opposite ray where $\arg(z) = \pi - \theta + 2m\pi$ we have $\arg(1/z) = -\pi + \theta + 2k\pi$.



2. Show that the map $f(z) = z + \frac{1}{z}$ maps the circle $|z| = r$ to an ellipse if $r \neq 1$ and to a line segment if $r = 1$. [Hint: Work in polar form, $z = r\cos\theta + ir\sin\theta$ and recall that an ellipse centered at the origin has an equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.]

   Solution: When $z = r\cos\theta + ir\sin\theta$ we have $1/z = (1/r)\cos(-\theta) + i(1/r)\sin(-\theta) = (1/r)\cos\theta - i(1/r)\sin\theta$ and

$$f(z) = z + \frac{1}{z} = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

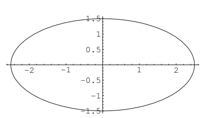
   If we fix r (so that z is on the circle of radius r about the origin) and let θ vary we get the image of the circle parametrically as

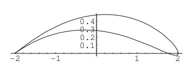
$$x = \left(r + \frac{1}{r}\right)\cos\theta \qquad y = \left(r - \frac{1}{r}\right)\sin\theta$$

   If $r = 1$ we get $x = 2\cos\theta$, $y = 0$ and so the image is the line segment $-2 \leq x \leq 2$ (because $\cos\theta$ can be any number in $[-1, 1]$).

If $r \neq 1$, then taking $a = r + 1/r$, $b = 1 - 1/r$ we get $x = a\cos\theta$, $y = b\sin\theta$, which is the usual parametrisation of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Here is a picture of the image for $r = 2$ and, as a point of interest we show the image of the circle $|z - (0.05 + 0.2i)| = |-1 - (0.05 + 0.2i)|$. This shape is known as a Joukowski aerofoil and the equations for 2-dimensional fluid flow past it can be solved by making use of the map we are considering here.





3. If $u(x, y) = (1/2)\ln(x^2 + y^2)$ and $v(x, y) = \tan^{-1}(y/x)$ for $x > 0$ show that u and v satisfy the Cauchy-Riemann equations.

Solution:

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{1}{2}\left(\frac{1}{x^2+y^2}\right)(2x) = \frac{x}{x^2+y^2} \\
\frac{\partial v}{\partial y} &= \frac{1}{1+(y/x)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{1+(y/x)^2}\frac{1}{x} \\
&= \frac{x}{x^2+y^2} = \frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y} &= \frac{1}{2}\left(\frac{1}{x^2+y^2}\right)(2y) = \frac{y}{x^2+y^2} \\
\frac{\partial v}{\partial x} &= \frac{1}{1+(y/x)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{1}{1+(y/x)^2}\frac{-y}{x^2} \\
&= -\frac{y}{x^2+y^2} = -\frac{\partial u}{\partial y}
\end{aligned}
$$

Note: This is actually quite a fundamental example. The corresponding analytic function $f(z) = f(x+iy) = u(x,y)+iv(x,y)$ (for $x = \Re z > 0$) is $f(z) = (1/2)\ln|z|^2+i\arg(z) = \ln|z| + i\arg(z)$. This is in fact the complex $\log z$ (or $\ln z$ if you prefer).

The formula $v(x,y) = \tan^{-1}(y/x)$ is going to run into trouble at the axis $x = 0$, but if we take it as $\arg(x+iy)$ we can cross the axis $x = 0$ except at the origin. We will in fact still have the Cauchy-Riemann equations satisfied as long as we don't go as far as the negative real axis where $\arg(z) = \pi$ but the principal value of $\arg(z)$ is discontinuous (changes suddenly to about $-\pi$ below the axis). There is no way to define $\log z$ (or $\arg z$) in a consistant continuous way for all complex $z \neq 0$. We always have to have a no-go line like the negative real axis.