3E1 Trinity Term Exercises
[May 7, 2003]

1. Show that the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n$$

has radius of convergence 1.

Solution: Use the ratio test. Limit of ratio of absolute values of $(n+1)$th term to nth

$$\lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+2}}{n+1}(z-1)^{n+1}\right|}{\left|\frac{(-1)^{n+1}}{n}(z-1)^n\right|} = \lim_{n\to\infty} \frac{n|z-1|}{n+1} = \lim_{n\to\infty} \frac{|z-1|}{1+1/n} = |z-1|$$

and so the series converges if $|z-1| < 1$ but fails to converge for $|z-1| > 1$. Thus the radius of convegence is $R = 1$.

2. Show that the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}(z-2)^n$$

has radius of convergence 2.

Solution: Ratio test again.

$$\lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+2}}{(n+1)2^{n+1}}(z-2)^{n+1}\right|}{\left|\frac{(-1)^{n+1}}{n2^n}(z-2)^n\right|} = \lim_{n\to\infty} \frac{n|z-2|}{2(n+1)} = \lim_{n\to\infty} \frac{|z-2|}{2(1+1/n)} = \frac{|z-2|}{2}$$

The series converges if $|z-2|/2 < 1$ (or $|z-2| < 2$) and fails to converge if $|z-2| > 2$. So the radius of convergence is $R = 2$.

3. If $f(z)$ denotes the sum of the power series in the first question (for $|z-1| < 1$) show that $f'(z) = 1/z$ for $|z-1| < 1$. [Note: Since $f(1) = 0 = \log 1$ and $f'(z) = \frac{d}{dz}\log z$ it follows that $\log z = f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n$ for $|z-1| < 1$.]

Solution: Because of the theorem that says we can differentiate power series ‘term by term’

$$f'(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} n(z-1)^{n-1} = \sum_{n=1}^{\infty} (1-z)^{n-1} = \sum_{n=0}^{\infty} (1-z)^n = \frac{1}{1-(1-z)} = \frac{1}{z}$$

(where we use the fact that the geometric series formula $\sum_{n=0}^{\infty} w^n = 1/(1-w)$ holds for $|w| < 1$).

4. Let $\gamma_r \colon [0, 2\pi] \to \mathbb{C}$ be the simple closed curve $\gamma_r(t) = 1 + re^{it}$ (circle of radius r about 1 once anticlockwise).

(a) Find

$$\int_{\gamma_r} \frac{1}{z-1}\,dz$$

by explicit calculation with the parametrisaion.

Solution: We replace z by $\gamma_r(t)$ and $dz = \gamma_r'(t)\,dt = ire^{it}\,dt$ to find

$$\int_{\gamma_r} \frac{1}{z-1}\,dz = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it}\,dt = \int_0^{2\pi} i\,dt = [it]_{t=0}^{2\pi} = 2\pi i$$

(b) Check that your answer agrees with the answer from Cauchy's integral formula.

Solution: Cauchy's integral formula applied with $f(z) = 1$ (which is analytic everywhere) and the anticlockwise simple closed curve γ_r (which has $z_0 = 1$ in its interior) tells us

$$\int_{\gamma_r} \frac{1}{z-1}\,dz = \int_{\gamma_r} \frac{f(z)}{z-1}\,dz = 2\pi i f(1) = 2\pi i$$

(c) If $r > 1$, show that

$$\int_{\gamma_r} \frac{\cos z}{(z-1)(z-2)}\,dz$$

does not depend on r.

Solution: The integrand $f(z) = \cos z/((z-1)(z-2))$ is analytic except at the two points $z = 1$ and $z = 2$. For $r > 1$ both of these points are inside γ_r (the circle of radius $r > 1$ about 1). So, for two different values of $r > 1$, say $1 < r_1 < r_2$, γ_{r_1} a simple closed curve that is inside the simple closed curve γ_{r_2}, both are anticlockwise and the integrand is analytic everywhere on and betwen the two curves. So

$$\int_{\gamma_{r_1}} \frac{\cos z}{(z-1)(z-2)}\,dz = \int_{\gamma_{r_2}} \frac{\cos z}{(z-1)(z-2)}\,dz$$

according to the (Corollary of) Cauchy's theorem.

(d) If $0 < r < 1$, find

$$\int_{\gamma_r} \frac{\cos z}{(z-1)(z-2)}\,dz$$

[Hint: $f(z) = \frac{\cos z}{z-2}$ is analytic for $|z| < 2$, which includes γ_r and its inside. Use Cauchy's integral formula.]

Solution: Using the hint, we can say

$$\int_{\gamma_r} \frac{\cos z}{(z-1)(z-2)}\,dz = \int_{\gamma_r} \frac{f(z)}{(z-1)}\,dz = 2\pi i f(1) = 2\pi i \frac{\cos 1}{-1} = -2\pi i \cos(1)$$