3E1 Trinity Term Tutorial sheet 10

[April 23 – 28, 2003]

Name: Solutions

1. Show that $\cos(z+w) = \cos z \cos w - \sin z \sin w$ (holds for $z, w \in \mathbb{C}$). [Hint: Write out both sides in terms of exponentials.]

   Solution:

$$\begin{aligned}
\cos(z+w) &= \frac{e^{i(z+w)} + e^{-i(z+w)}}{2} \\
&= \frac{e^{iz}e^{iw} + e^{-iz}e^{-iw}}{2} \\
\cos z \cos w - \sin z \sin w &= \frac{e^{iz}+e^{-iz}}{2}\frac{e^{iw}+e^{-iw}}{2} - \frac{e^{iz}-e^{-iz}}{2i}\frac{e^{iw}-e^{-iw}}{2i} \\
&= \frac{e^{iz}e^{iw} + e^{-iz}e^{-iw} + e^{-iz}e^{iw} + e^{iz}e^{-iw}}{4} \\
&\quad - \frac{e^{iz}e^{iw} + e^{-iz}e^{-iw} - e^{-iz}e^{iw} - e^{iz}e^{-iw}}{-4} \\
&= \frac{e^{iz}e^{iw} + e^{-iz}e^{-iw}}{2} \\
&= \cos(z+w)
\end{aligned}$$

2. Find the real and imaginary parts of $\cos(2+3i)$

   Solution:

$$\begin{aligned}
\cos(2+3i) &= \cos 2 \cos(3i) - \sin 2 \sin(3i) \\
&= \cos 2 \cosh 3 - i \sin 2 \sinh 3
\end{aligned}$$

3. Find z with $\sin z = 5$. [Hint: Solve a quadratic for e^{iz}]

   Solution: $\sin z = (e^{iz} - e^{-iz})/(2i)$ and so we want

$$\begin{aligned}
e^{iz} - e^{-iz} &= 5(2i) = 10i \\
&\qquad \text{Multiply across by } e^{iz} \\
(e^{iz})^2 - 1 &= 10ie^{iz} \\
(e^{iz})^2 - 10ie^{iz} - 1 &= 0 \\
e^{iz} &= (1/2)(-b \pm \sqrt{b^2 - 4ac}) \\
&= (1/2)(10i \pm \sqrt{-100+4}) \\
&= (1/2)(10i \pm i\sqrt{96}) \\
&= (5 \pm \sqrt{24})i \\
iz &= \ln(5+\sqrt{24}) + i\pi/2 \\
&\quad \text{is one possible solution} \\
z &= \pi/2 - i\ln(5+\sqrt{24})
\end{aligned}$$

4. Find (the real and imaginary parts of) $\log(2-2i)$

   Solution:

$$\begin{aligned}\log(2-2i) &= \ln|2-2i| + i\arg(2-2i)\\ &= \ln\sqrt{8} + i\tan^{-1}(-1)\\ &= \frac{1}{2}\ln 8 - i\frac{\pi}{4}\end{aligned}$$

Exercises

1. Show that the power series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n$$

   has radius of convergence 1.

2. Show that the power series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n2^n}(z-2)^n$$

   has radius of convergence 2.

3. If $f(z)$ denotes the sum of the power series in the first question (for $|z-1|<1$) show that $f'(z)=1/z$ for $|z-1|<1$. [Note: Since $f(1)=0=\log 1$ and $f'(z)=\frac{d}{dz}\log z$ it follows that $\log z = f(z) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n$ for $|z-1|<1$.]

4. Let $\gamma_r\colon [0,2\pi]\to\mathbb{C}$ be the simple closed curve $\gamma_r(t)=1+re^{it}$ (circle of radius r about 1 once anticlockwise).

   (a) Find

$$\int_{\gamma_r}\frac{1}{z-1}\,dz$$

   by explicit calculation with the parametrisaion.

   (b) Check that your answer agrees with the answer from Cauchy's integral formula.

   (c) If $r>1$, show that

$$\int_{\gamma_r}\frac{\cos z}{(z-1)(z-2)}\,dz$$

   does not depend on r.

   (d) If $0<r<1$, find

$$\int_{\gamma_r}\frac{\cos z}{(z-1)(z-2)}\,dz$$

   [Hint: $f(z)=\frac{\cos z}{z-2}$ is analytic for $|z|<2$, which includes γ_r and its inside. Use Cauchy's integral formula.]